Introduction to using pressure as a vertical coordinate

When the hydrostatic approximation is used, it often makes sense to change vertical coordinate from height (z) to pressure (p). The principal advantages of the pressure coordinate are that the mass continuity equation does not contain a time derivative, and the pressure force term in the equation for the horizontal velocity does not involve the density. The principal disadvantages are that the value of the vertical coordinate at the lower boundary is no longer fixed in time[1], and that the static stability parameter is strongly varying in the vertical.

Note that the unit vector in the vertical remains in the same direction when we change vertical coordinate from z to p. However, partial derivatives in the horizontal such as $\partial T/\partial x$ change because we are holding p rather than z constant and surfaces of constant p are generally tilted with respect to surfaces of constant z.

These notes broadly follow section 2.6.2 of the Vallis textbook (2nd edition). See also sections 1.4.2 and 3.1 of Holton and Hakim.

Vertical velocity

The vertical velocity in pressure coordinates is given by $\omega = Dp/Dt$. This is analogous to how we define the horizontal velocities ($u = Dx/Dt$ and $v = Dy/Dt$) or the vertical velocity in height coordinates ($w = Dz/Dt$).

Lagrangian derivative

The Lagrangian derivative is expressed in pressure coordinates as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \omega \frac{\partial}{\partial p}, \tag{1}$$

where $\mathbf{u} = (u, v)$. Both the horizontal gradient ∇ and the time derivative $(\partial/\partial t)$ are taken at constant p rather than z.

Mass continuity equation

Mass conservation for a material element of air may be written as

$$\frac{D\rho\delta V}{Dt} = 0, \tag{2}$$

where $\delta V = \delta x \delta y \delta z$ is the volume of the material element. Hydrostatic balance gives us that $\rho \delta z = -\delta p/g$, such that

$$\frac{D\delta x \delta y \delta p}{Dt} = 0. \tag{3}$$

[1]A common compromise in numerical modeling is to use a terrain following coordinate such as $\sigma = p/p_s$ where p_s is surface pressure.

We then use that $D\delta x/Dt = \delta u$ where δu is the change in u across the material element in the x direction. Similarly $D\delta y/Dt = \delta v$ and $D\delta p/Dt = \delta\omega$. Substituting into equation 3 gives that

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta\omega}{\delta p} = 0. \tag{4}$$

In the limit of an infinitesimal parcel of air, this then gives the mass continuity equation in pressure coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} = 0. \tag{5}$$

Pressure force in the horizontal

The pressure force term in the horizontal momentum equation in z coordinates may be written as

$$-\frac{1}{\rho}\left(\nabla p\right)_z, \tag{6}$$

where the subscript z make explicit that horizontal derivatives are taken at constant z. To convert this to pressure coordinates, we first write the general rule for converting a derivative with respect to x from the z vertical coordinate to the p vertical coordinate:

$$\left(\frac{\partial}{\partial x}\right)_p = \left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial}{\partial z}. \tag{7}$$

Applying this rule to the derivative of p gives that

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}. \tag{8}$$

Using hydrostatic balance then gives that

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial\phi}{\partial x}\right)_p, \tag{9}$$

where $\phi = gz$ is the geopotential. Finally, considering derivatives with respect to both x and y gives that

$$-\frac{1}{\rho}\left(\nabla p\right)_z = -\left(\nabla\phi\right)_p. \tag{10}$$

Hydrostatic balance (vertical momentum equation)

Hydrostatic balance in z coordinates ($\partial p/\partial z = -\rho g$) is more conveniently written using the ideal gas law as

$$\frac{\partial\phi}{\partial p} = -\frac{RT}{p}. \tag{11}$$

Thermodynamic equation and static stability parameter

The thermodynamic equation in the absence of diabatic heating and written in terms of potential temperature (θ) remains:

$$\frac{D\theta}{Dt} = 0. \tag{12}$$

Alternatively it can be written in terms of temperature T as

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \omega S_p = 0, \tag{13}$$

where the static stability parameter is

$$S_p = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}. \tag{14}$$

One disadvantage of pressure coordinates is that S_p increases with height, whereas static stability in z coordinates is usually expressed as $N^2 = g(\partial\theta/\partial z)/\theta$ which is relatively constant over the troposphere.

Summary

The equations for horizontal velocity, hydrostatic balance, mass continuity and potential temperature in pressure coordinates in the absence of friction, diabatic heating, and planetary rotation are:

$$\frac{D\mathbf{u}}{Dt} = -\nabla\phi, \tag{15}$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p}, \tag{16}$$

$$\nabla \cdot \mathbf{u} + \frac{\partial \omega}{\partial p} = 0, \tag{17}$$

$$\frac{D\theta}{Dt} = 0. \tag{18}$$