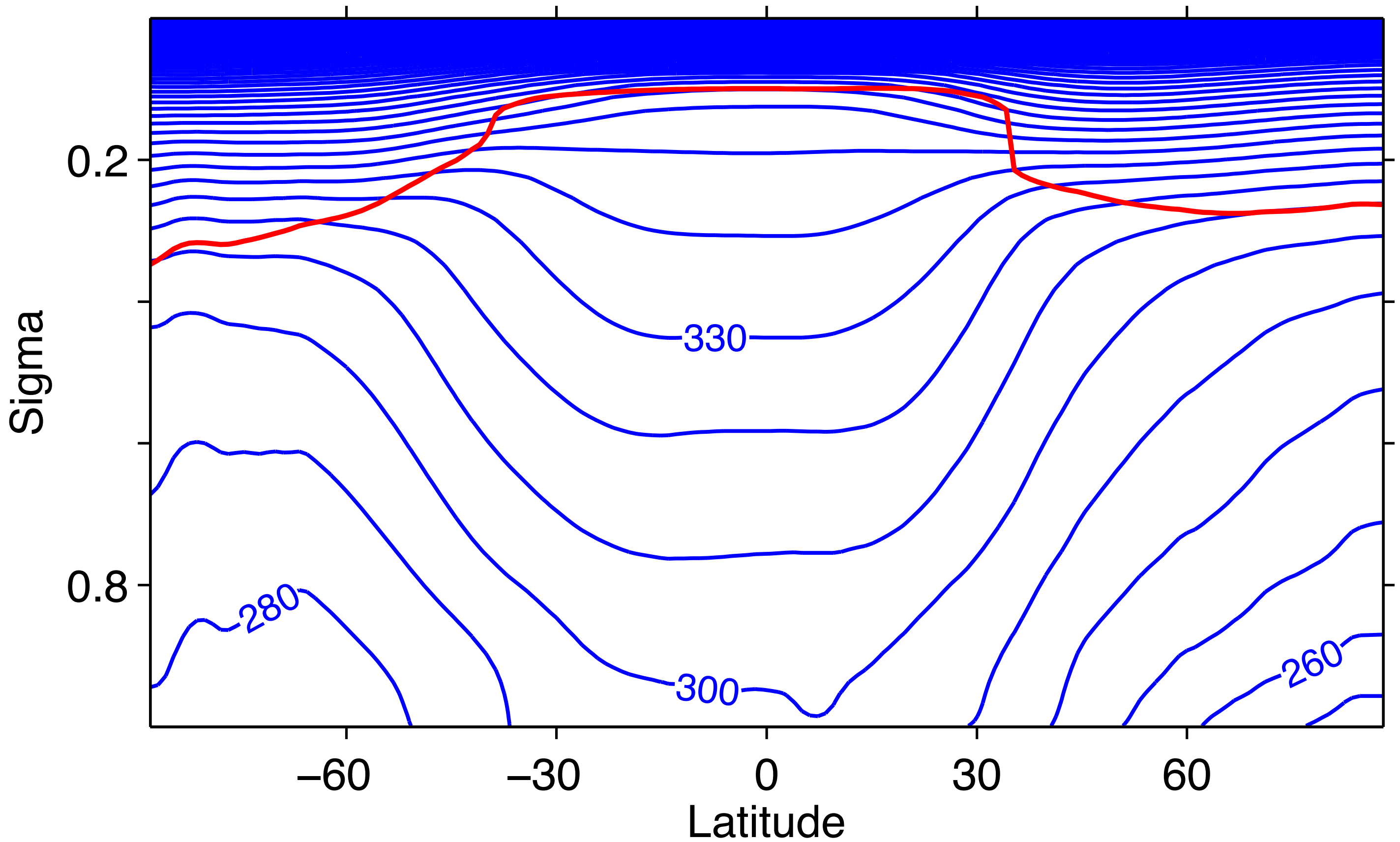
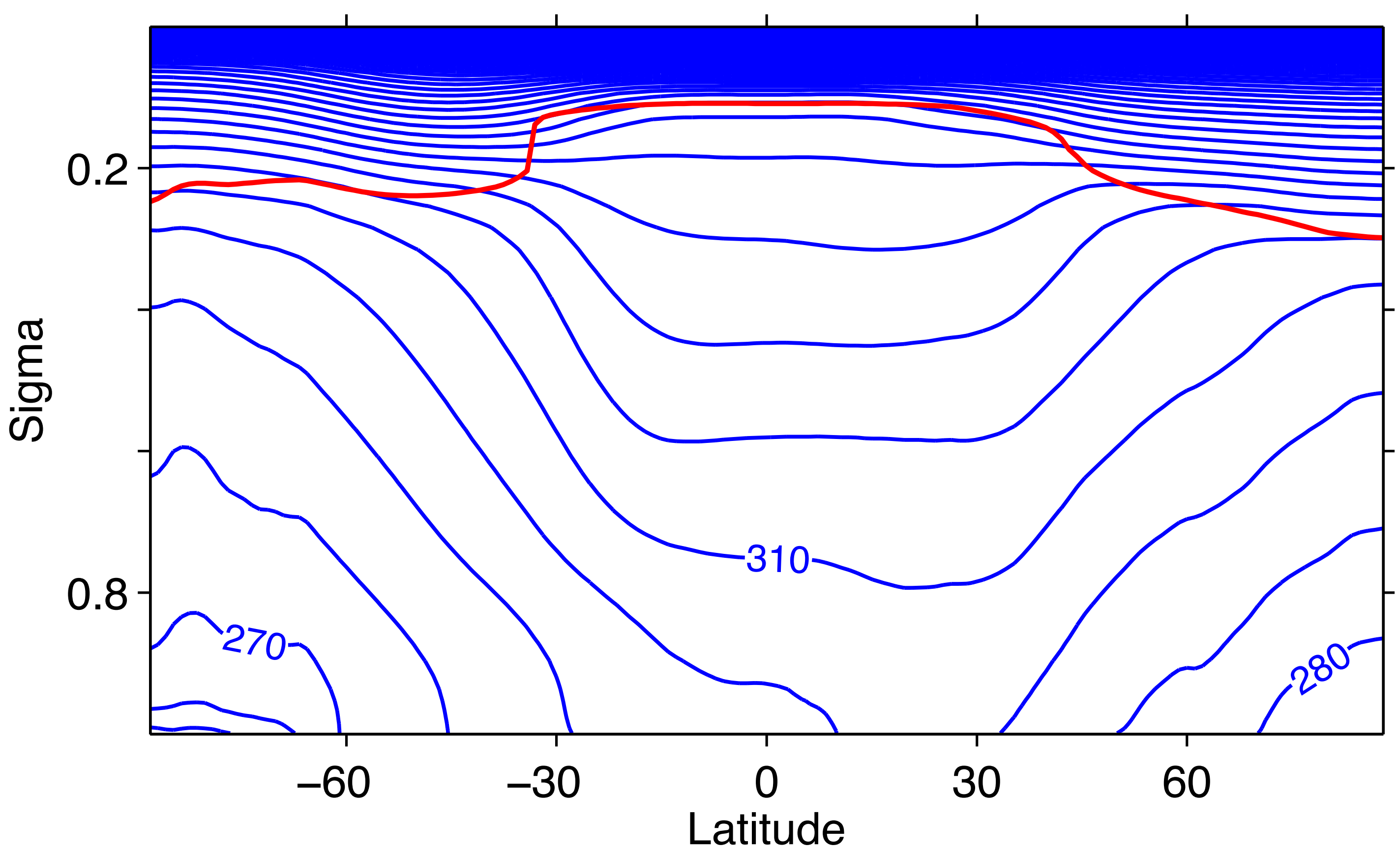
12.810 Dynamics of the Atmosphere

Internal gravity waves in the atmosphere



DJF



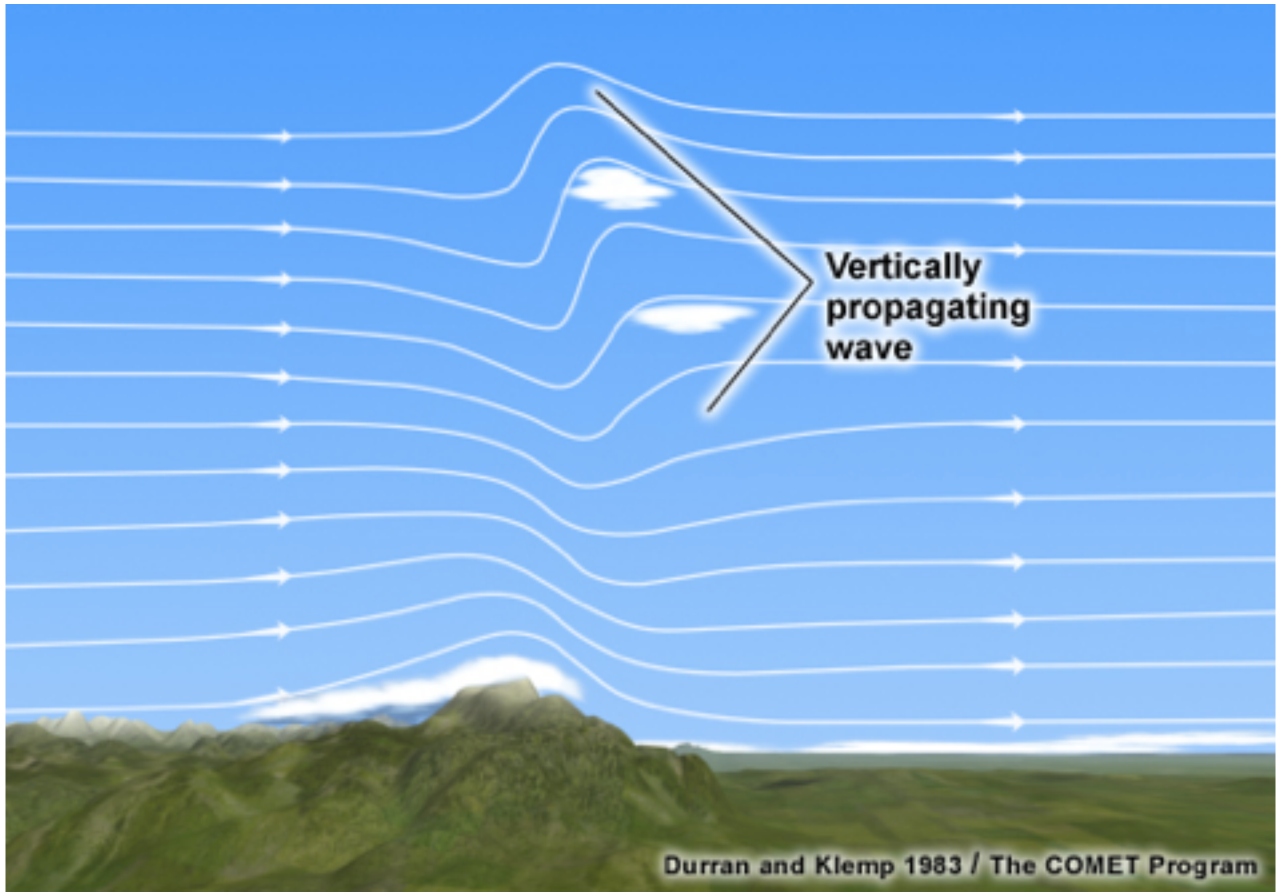
JJA

Potential temperature (K)

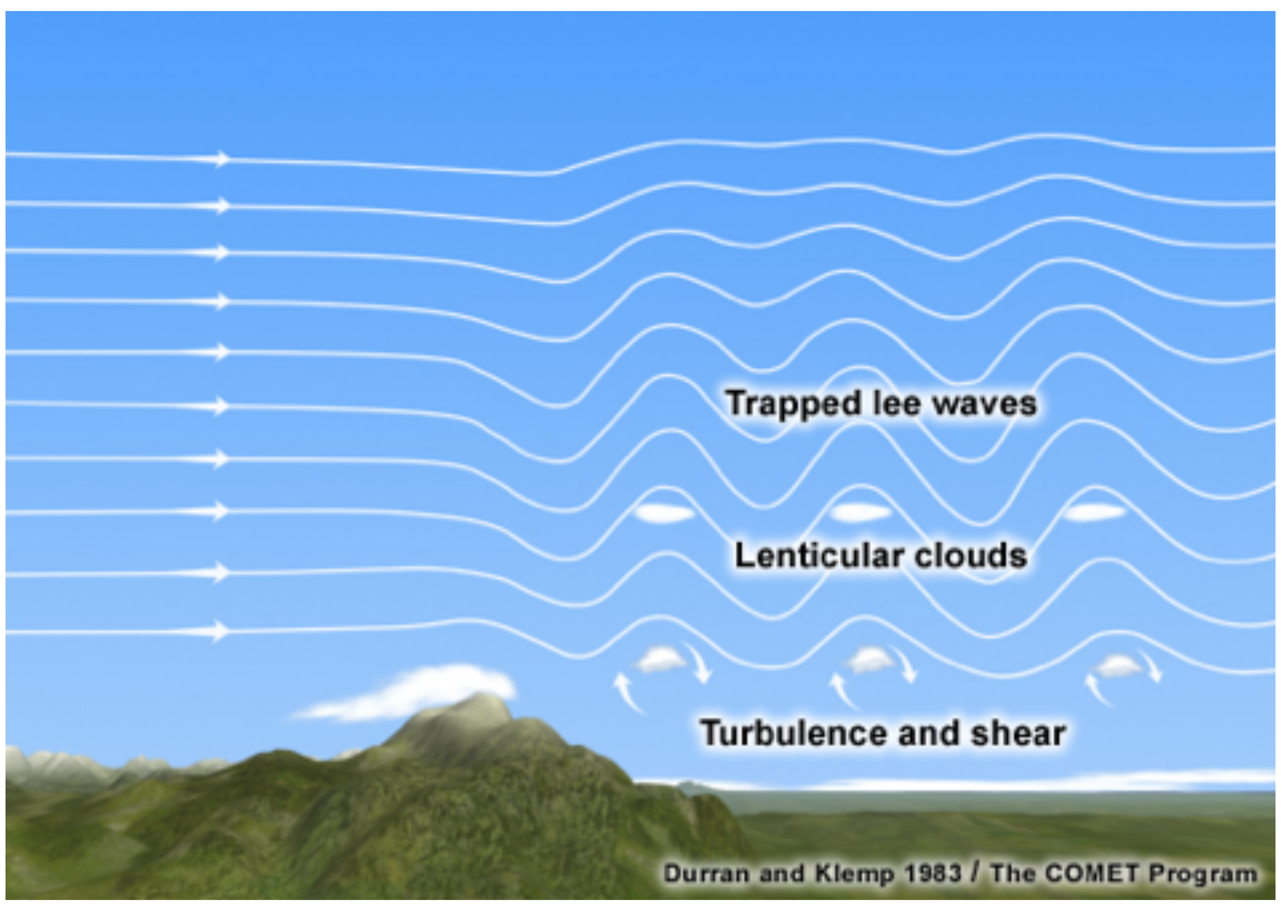
Increase with height (implies dry static stability)

(ERA40 reanalysis data 1980-2001)

Positive static stability allows for internal gravity waves: here forced by mountain



Vertically propagating

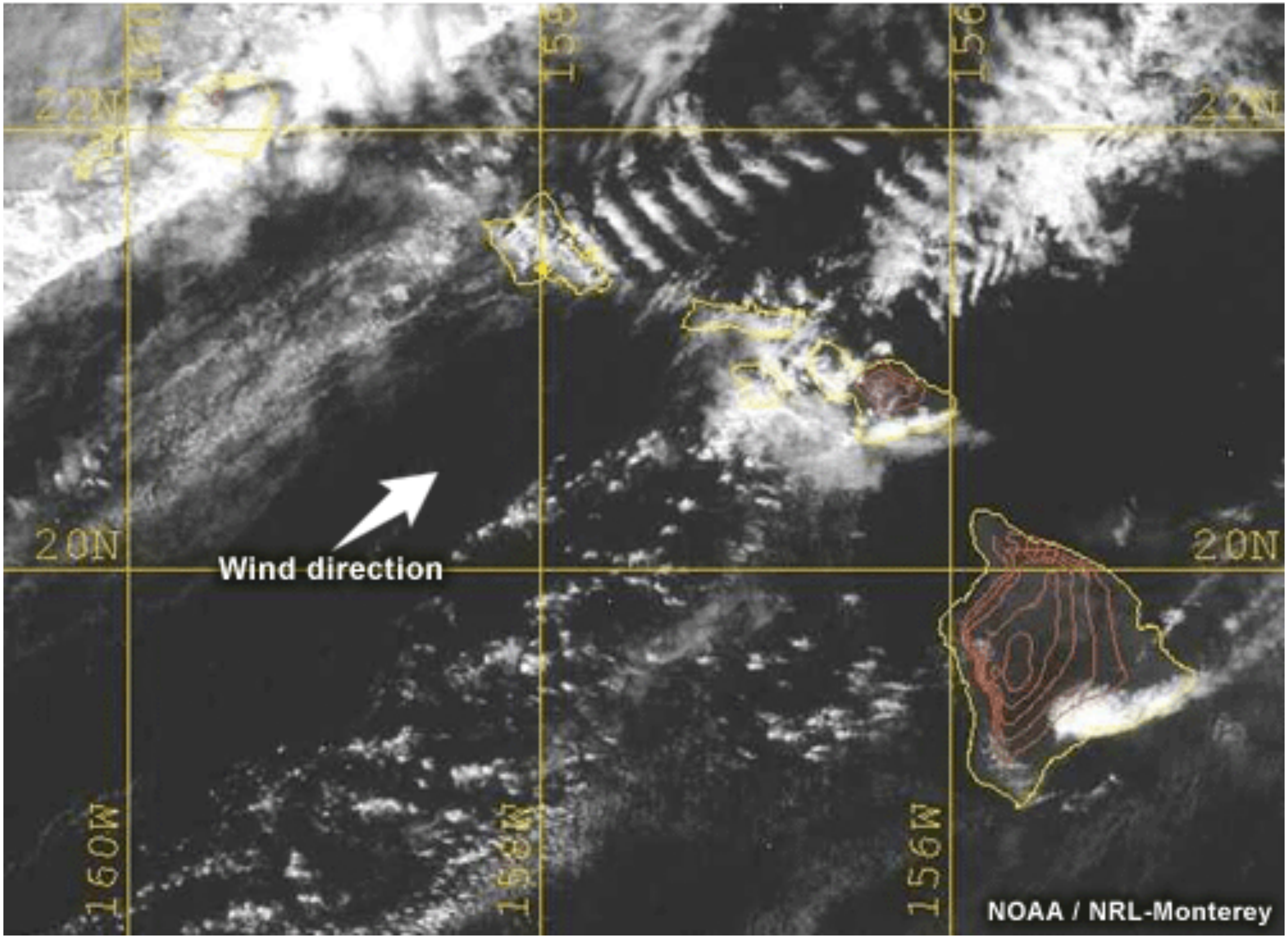


Trapped

Also forced by moist convection, geostrophic adjustment, surface warming/cooling...

Trapped lee waves downwind from Hawaiian Islands

GOES-10 VIS Image 2000 UTC 24 Jan 2003



Internal gravity waves

- Basic theory of internal gravity waves will first be introduced (see handout)

- Then discuss:

  - mountain waves

  - compressible gravity waves and vertical propagation

  - interaction of gravity waves with mean flow

Internal gravity waves: Introductory material

Governing equtions for non-rotating, inviscid, adiabatic flow in Boussinesq approximation

$$\frac{Du}{Dt} = -\frac{\partial \phi}{\partial x}$$

$$\frac{Dv}{Dt} = -\frac{\partial \phi}{\partial y}$$

$$\frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} + b$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{Db}{Dt} = 0$$

Rewrite for convenience

$$\alpha \frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} + b$$

α=1: full equations
α=0: hydrostatic

Waves on a basic state

- Basic state is an exact solution on which waves propagate
- Choose a basic state that is at rest and stably stratified:

$$
\begin{aligned}
u_0 &= v_0 = w_0 = 0 \\
b_0 &= N^2 z \\
\phi_0 &= \int b_0 \, dz
\end{aligned}
$$

- N is the buoyancy frequency (the angular frequency at which a parcel moving vertically would oscillate)
- Full solution is basic state plus a perturbation that is the wave. For example, for buoyancy:

$$b = b_0 + b'$$

Assume small amplitude perturbations and linearize the equations (drop terms that are squared in wave amplitude)

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -\frac{\partial \phi'}{\partial y}$$

$$\alpha \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b'$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial b'}{\partial t} + N^2 w' = 0$$

Look for wavelike solutions

$$\begin{pmatrix} u' \\ v' \\ w' \\ \phi' \\ b' \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} U \\ V \\ W \\ \Phi \\ B \end{pmatrix} e^{i(kx+ly+mz-\omega t)}\right]$$

where **k**=(k,l,m) is the wavenumber vector and ω is the angular frequency

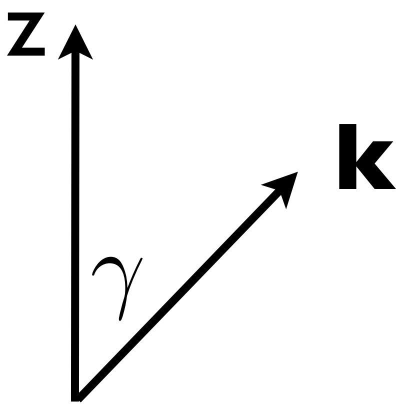
$$\begin{aligned} \omega U - k\Phi &= 0 \\ \omega V - l\Phi &= 0 \\ \omega\alpha W - m\Phi - iB &= 0 \\ kU + lV + mW &= 0 \\ -i\omega B + N^2 W &= 0 \end{aligned}$$

Dispersion relation for non-hydrostatic (α=1) waves

$$\omega = \pm N \sqrt{\frac{k^2 + l^2}{k^2 + l^2 + m^2}}$$

Or put more simply

$$\omega = \pm N \, \sin\gamma$$



Which implies that

$$|\omega| \leq N$$ (no propagation otherwise!)

Propagation: Phase and group velocities

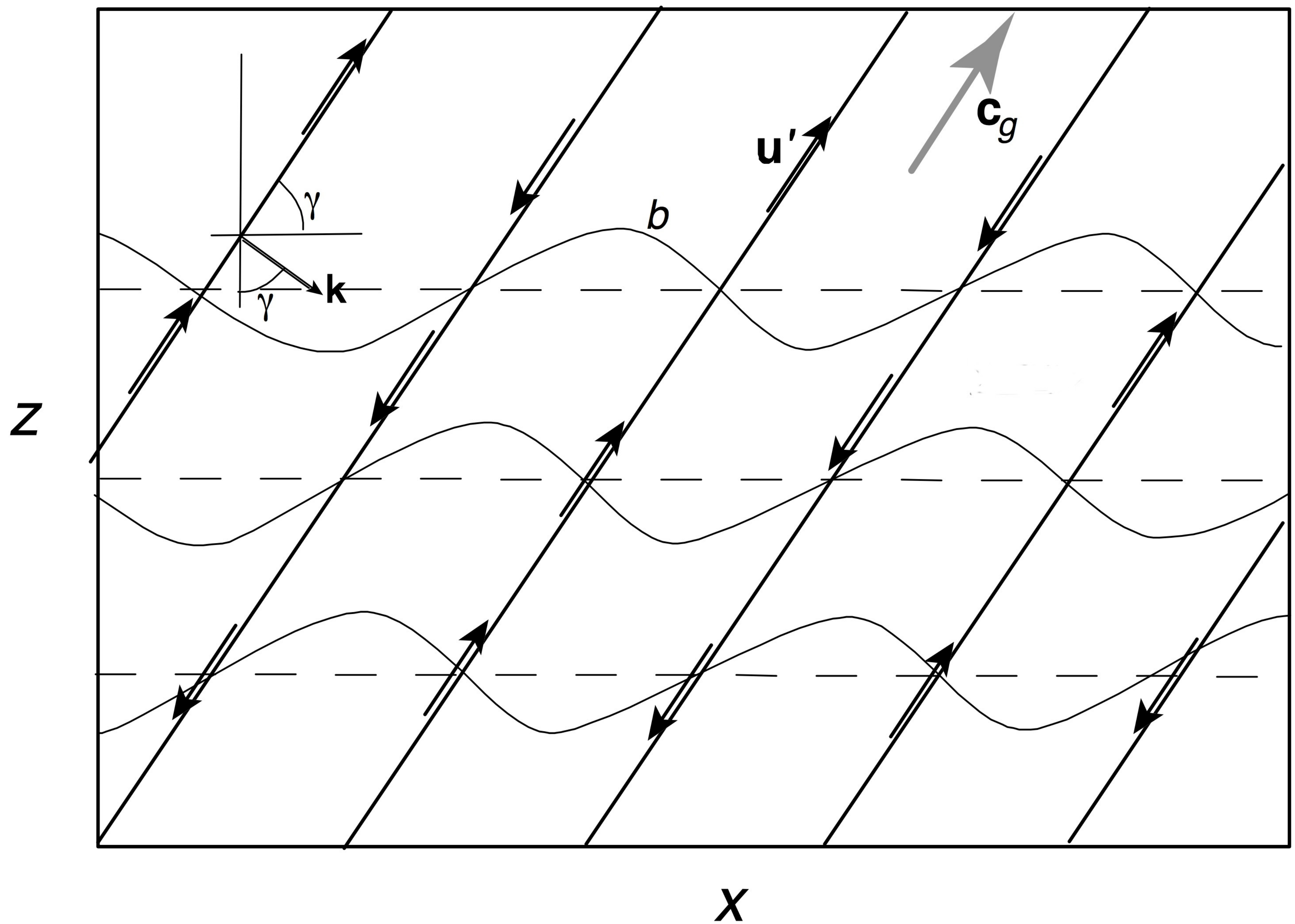
The phase speed in the direction of $\mathbf{k}$ is given by

$$c = \frac{\omega}{|\mathbf{k}|}$$

and the group velocity is

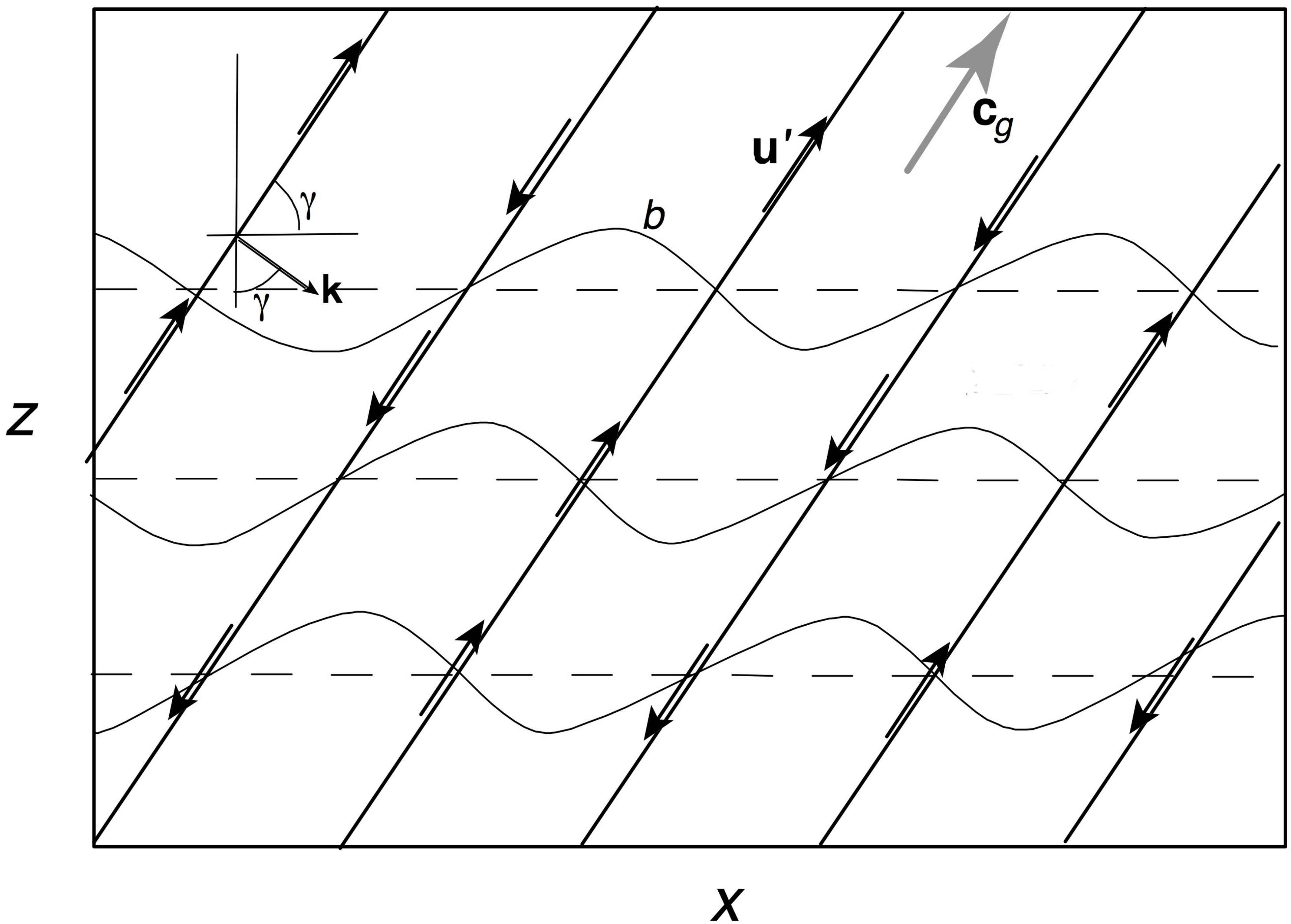
$$\mathbf{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) = \frac{\omega m}{(k^2 + l^2 + m^2)} \left[\frac{km}{k^2 + l^2}, \frac{lm}{k^2 + l^2}, -1 \right]$$

A wave with group velocity upwards and to the right

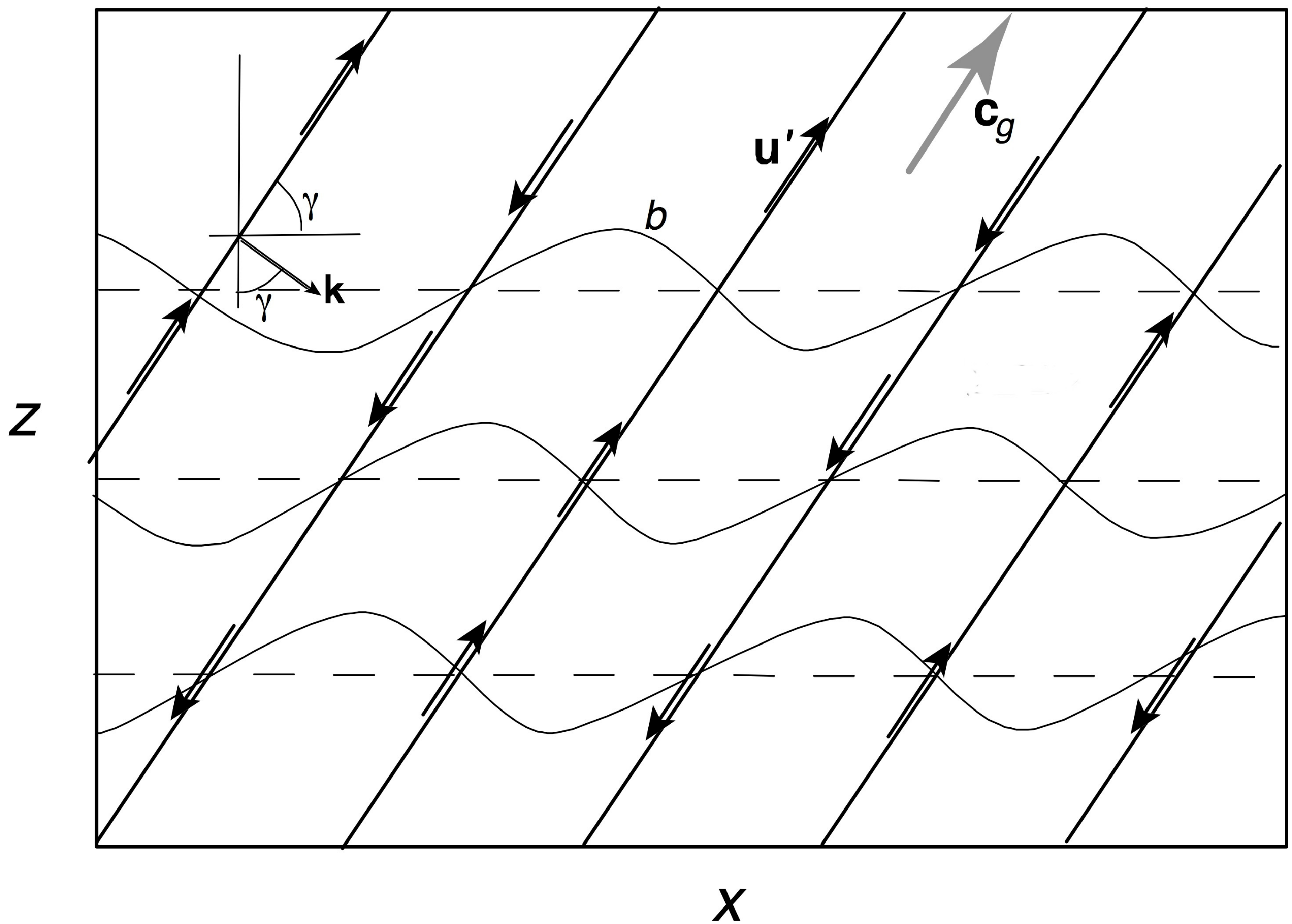


Wavy lines are isolines of $b=b'+b_0(z)$
Black arrows show velocity

$\mathbf{c_g \cdot k} = 0$: Group propagation is along phase lines!

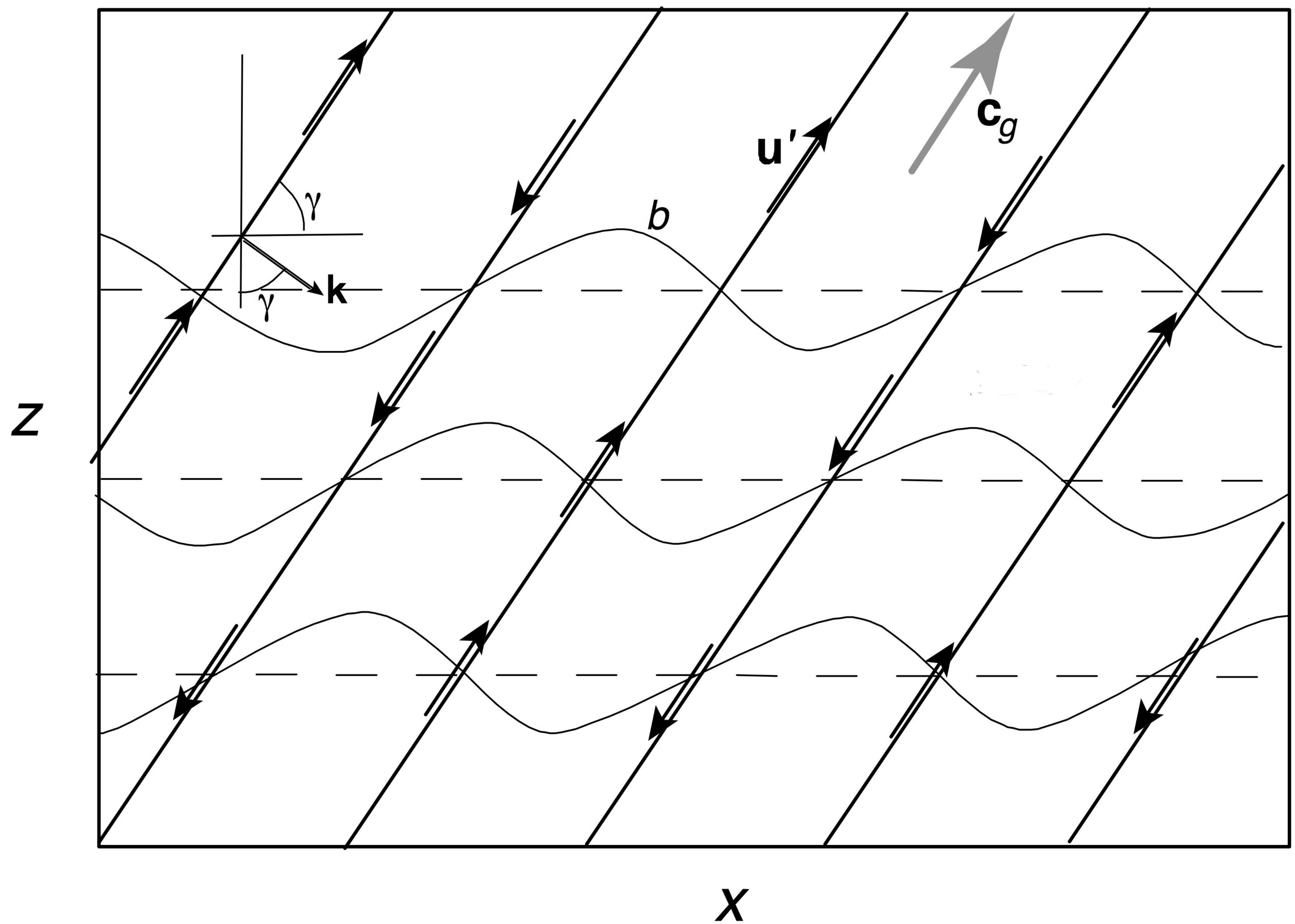


Group velocity is upwards if phase propagation downwards! (but both phase and group propagate to the right)



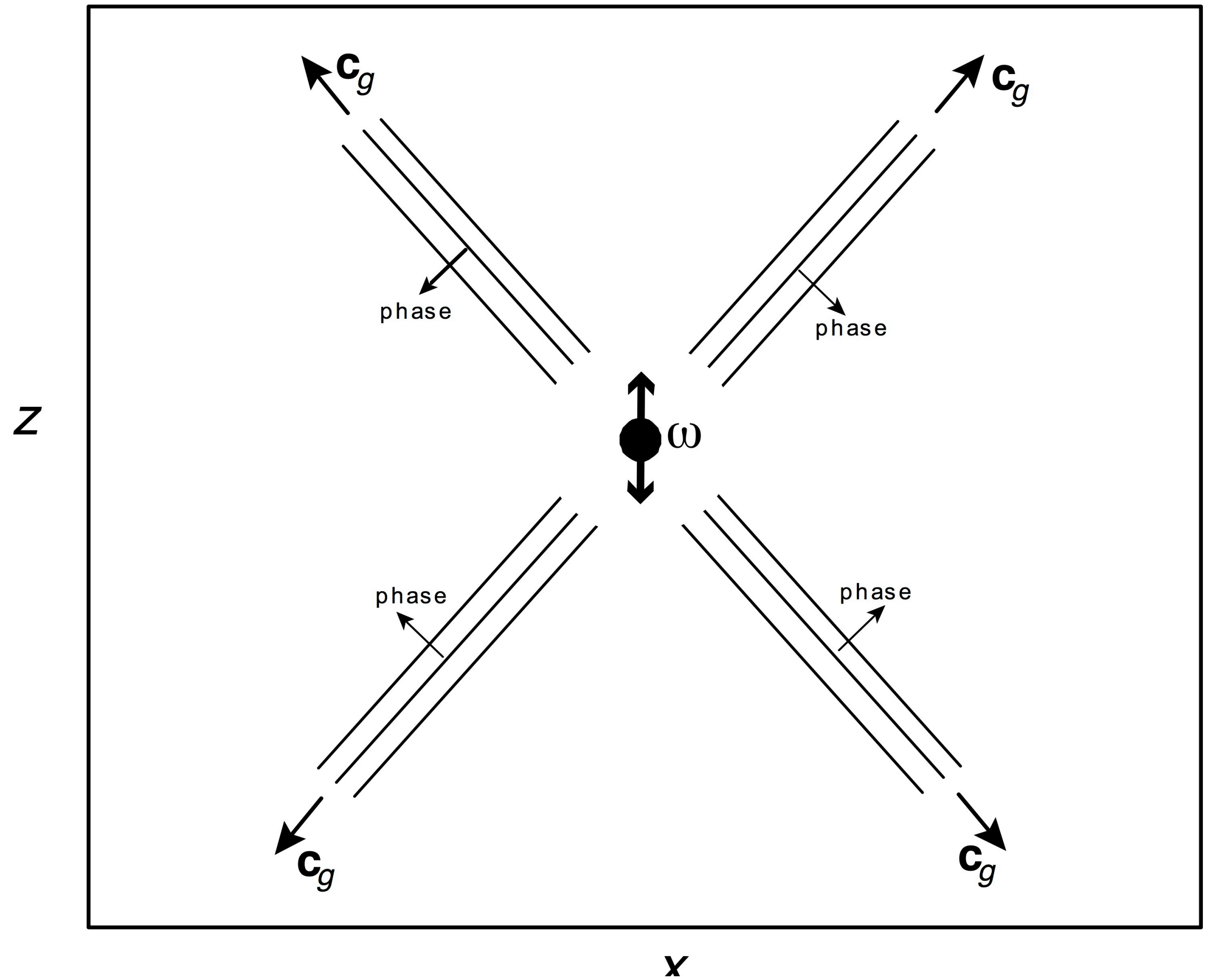
z direction is special because of gravity

k•**u** = 0: Fluid motions are along phase lines

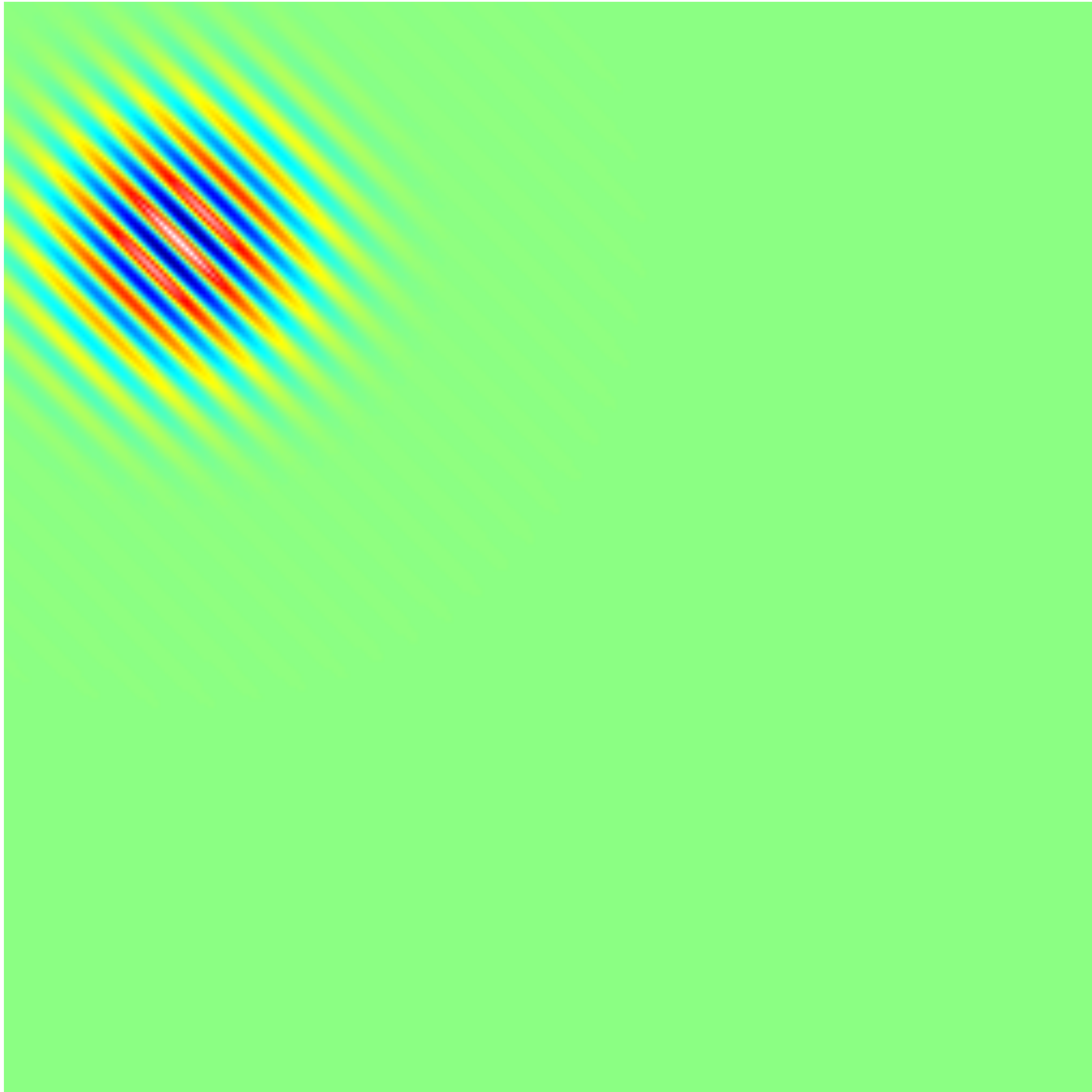


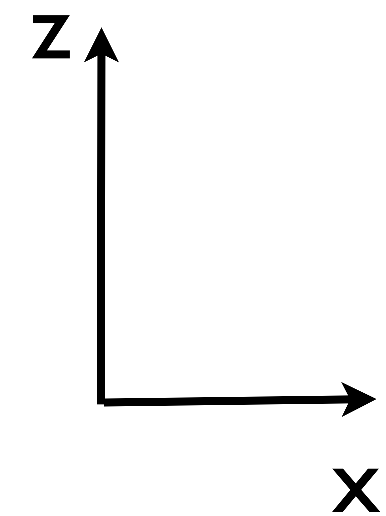
Implies that no advection of wave properties such as b:
plane gravity wave is a nonlinear solution!

From a localized source oscillating with a single frequency ω, the waves form rays (the "St Andrews' cross") at angles $\gamma = \sin^{-1}(\omega/N)$ to the horizontal, with the phase propagation *across* the rays:



Animation courtesy
Glenn Flierl

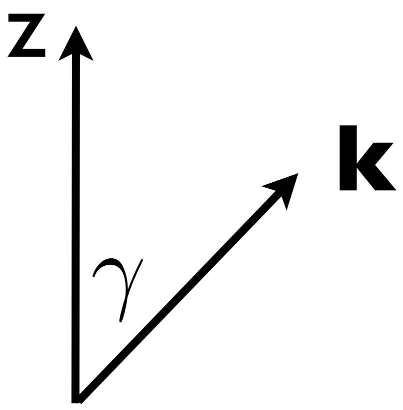




$\mathbf{k}=(k,m)=(2,2)$; $\mathbf{c}_g=(0.18,-0.18)$

Relation of frequency to buoyancy frequency N

$$\omega = \pm N \ \sin\gamma$$



Implications of **k•u** = 0

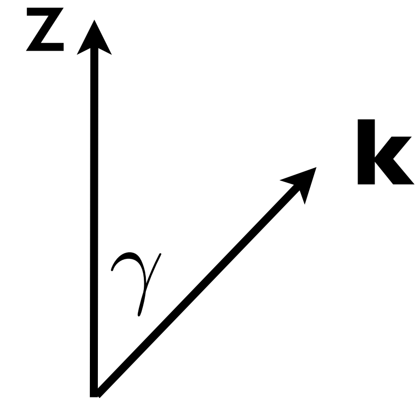
(fluid motions perpendicular to wavevector)

$\gamma \rightarrow \pi/2$ motions are vertical and $\omega \rightarrow N$

$\gamma \rightarrow 0$ motions are horizontal $\omega \rightarrow 0$

Relation of frequency to buoyancy frequency N

$$\omega = \pm N \ \sin\gamma$$



Implications of **k·u** = 0

(fluid motions perpendicular to wavevector)

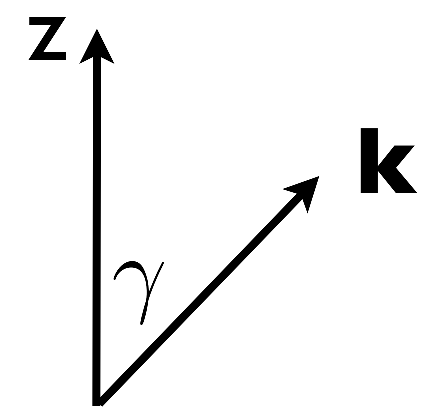
$\gamma \to \pi/2$ motions are vertical and $\omega \to N$

$\gamma \to 0$ motions are horizontal $\omega \to 0$

No resistance from stratification!

Hydrostatic case (set α=0)

$$\omega = \pm \frac{N}{m}\sqrt{k^2+l^2} = \pm N \tan\gamma$$



Only a good approximation to

$$\omega = \pm N \ \sin\gamma$$

when γ is small i.e. $k^2+l^2 \ll m^2$

This is true when vertical length scales are small compared to horizontal length scales

Mountain waves

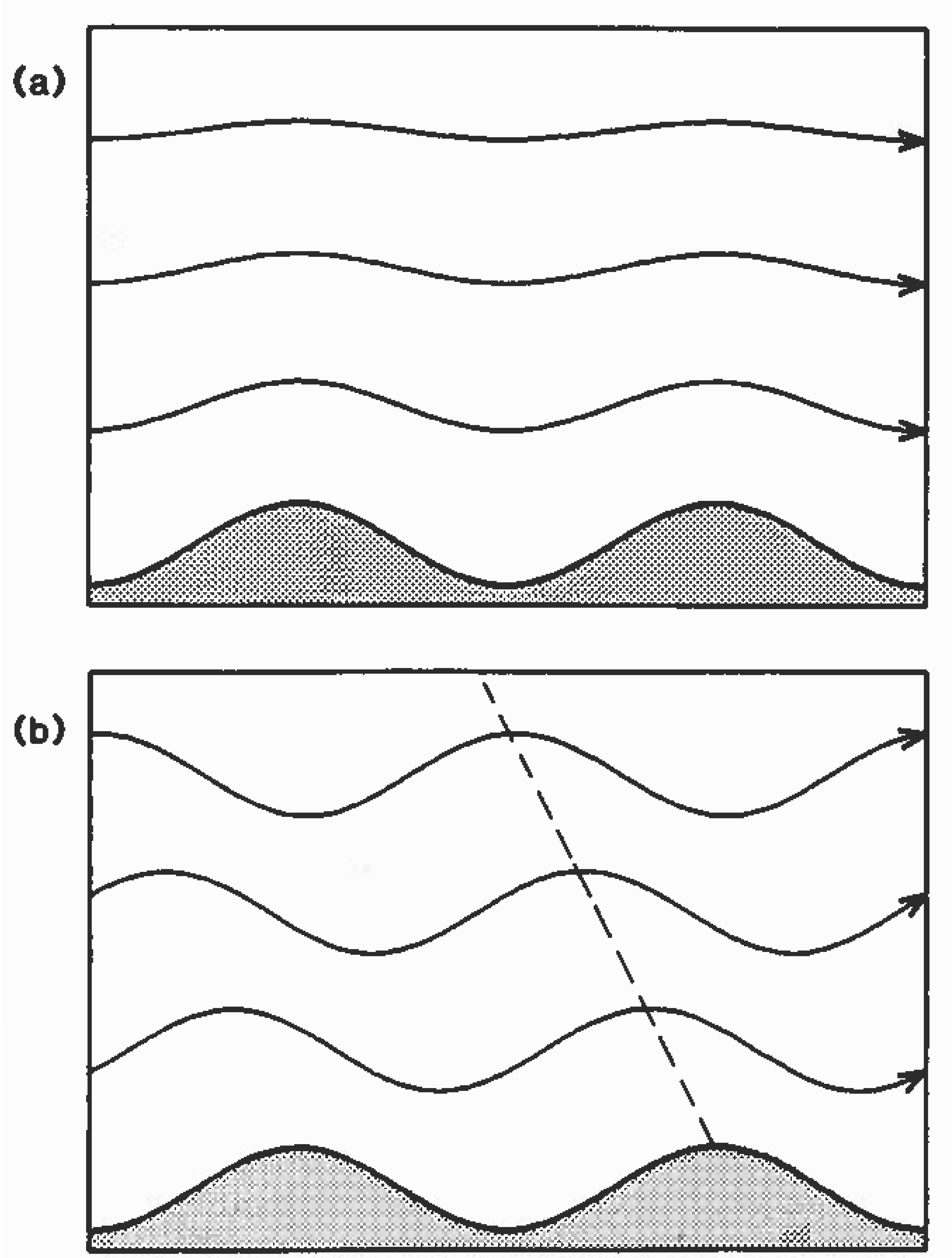
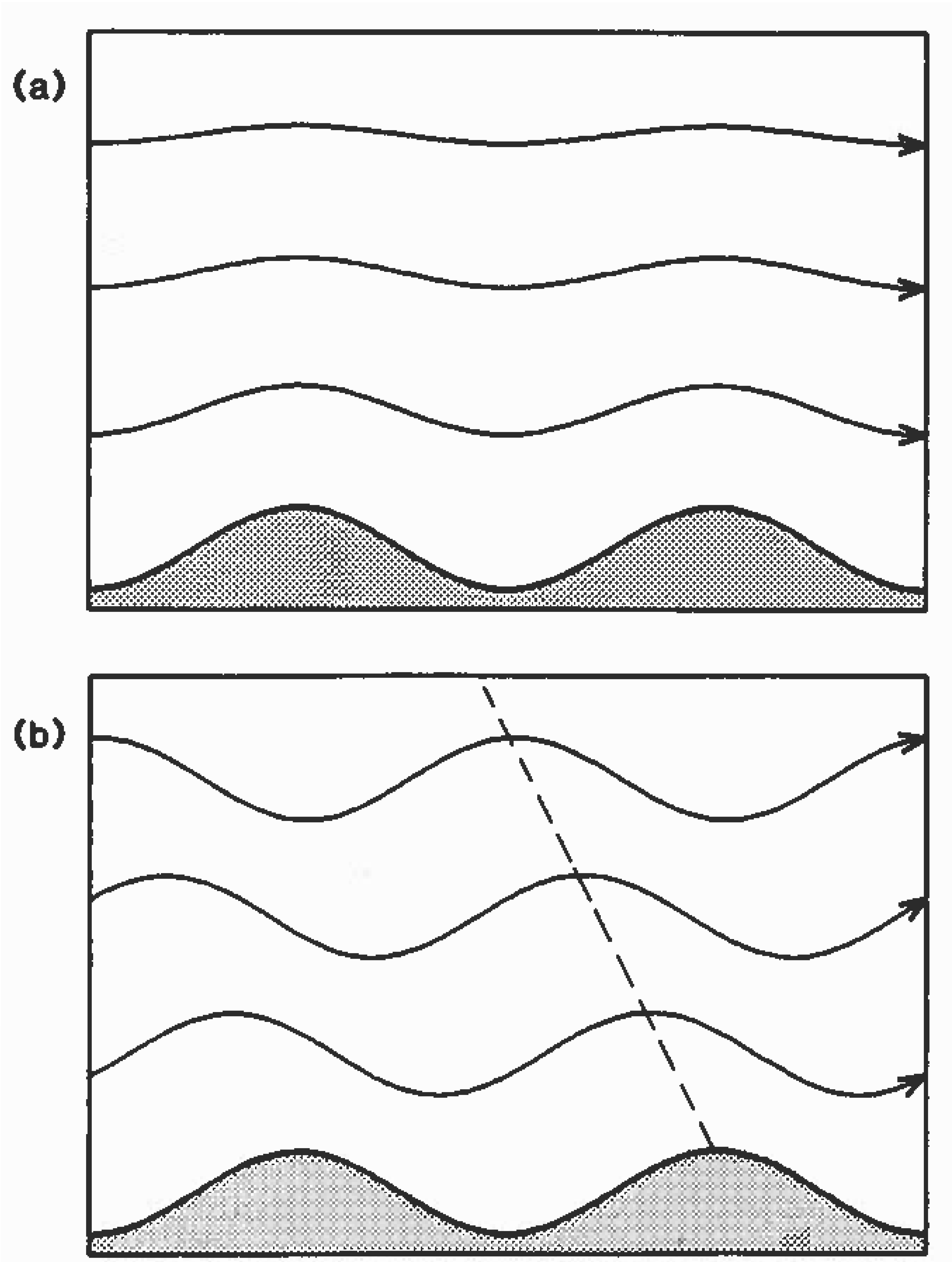


Fig 1 Streamlines over periodic mountains

Durran, AMS, 1990

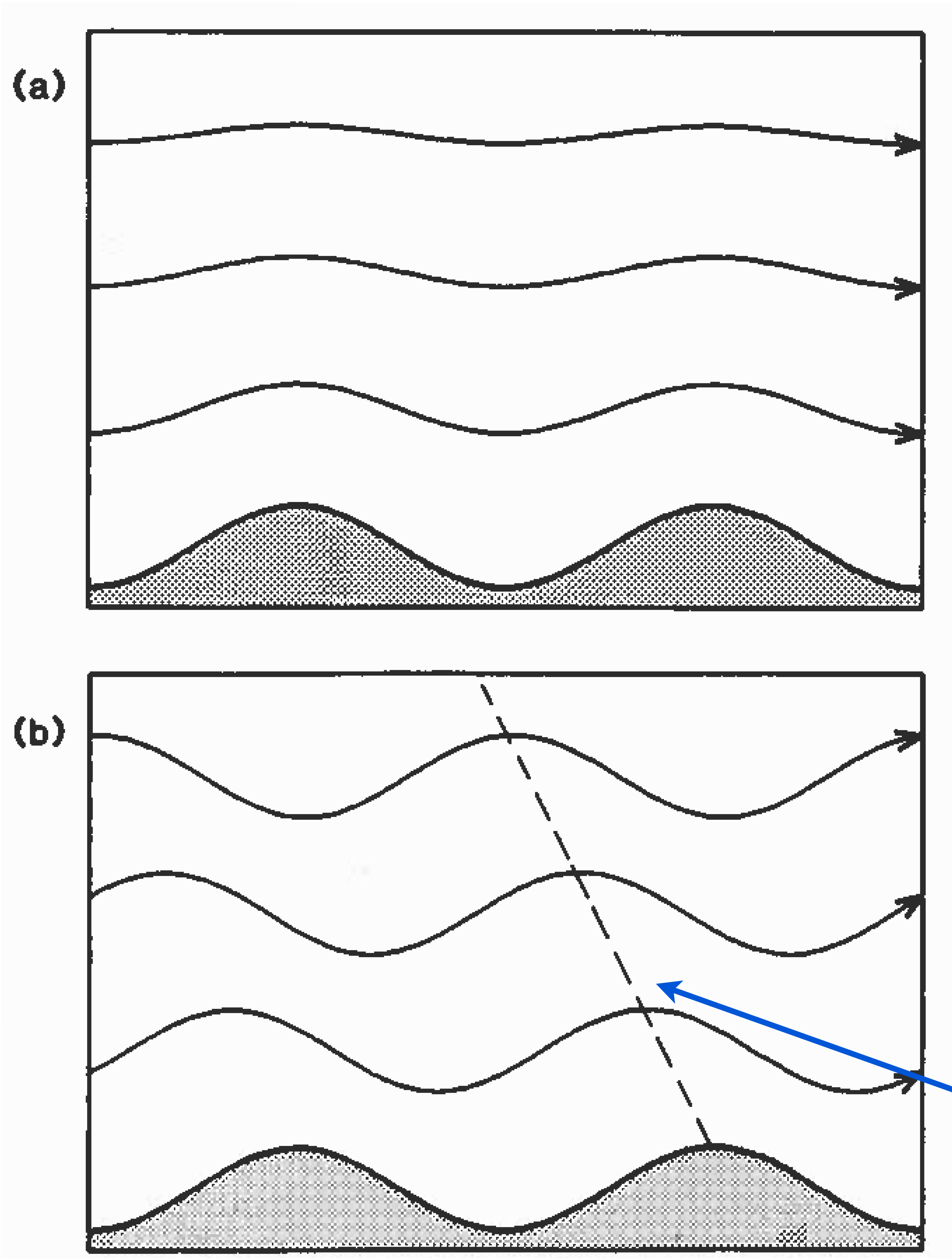


Evanescent waves
(e.g. weak stratification)

Vertically propagating waves
(e.g. strong stratification)

Fig 1 Streamlines over periodic mountains

Durran, AMS, 1990



Evanescent waves
(e.g. weak stratification)

No phase tilt with height

Vertically propagating waves
(e.g. strong stratification)

Phase tilt with height

Fig 1 Streamlines over periodic mountains

Durran, AMS, 1990

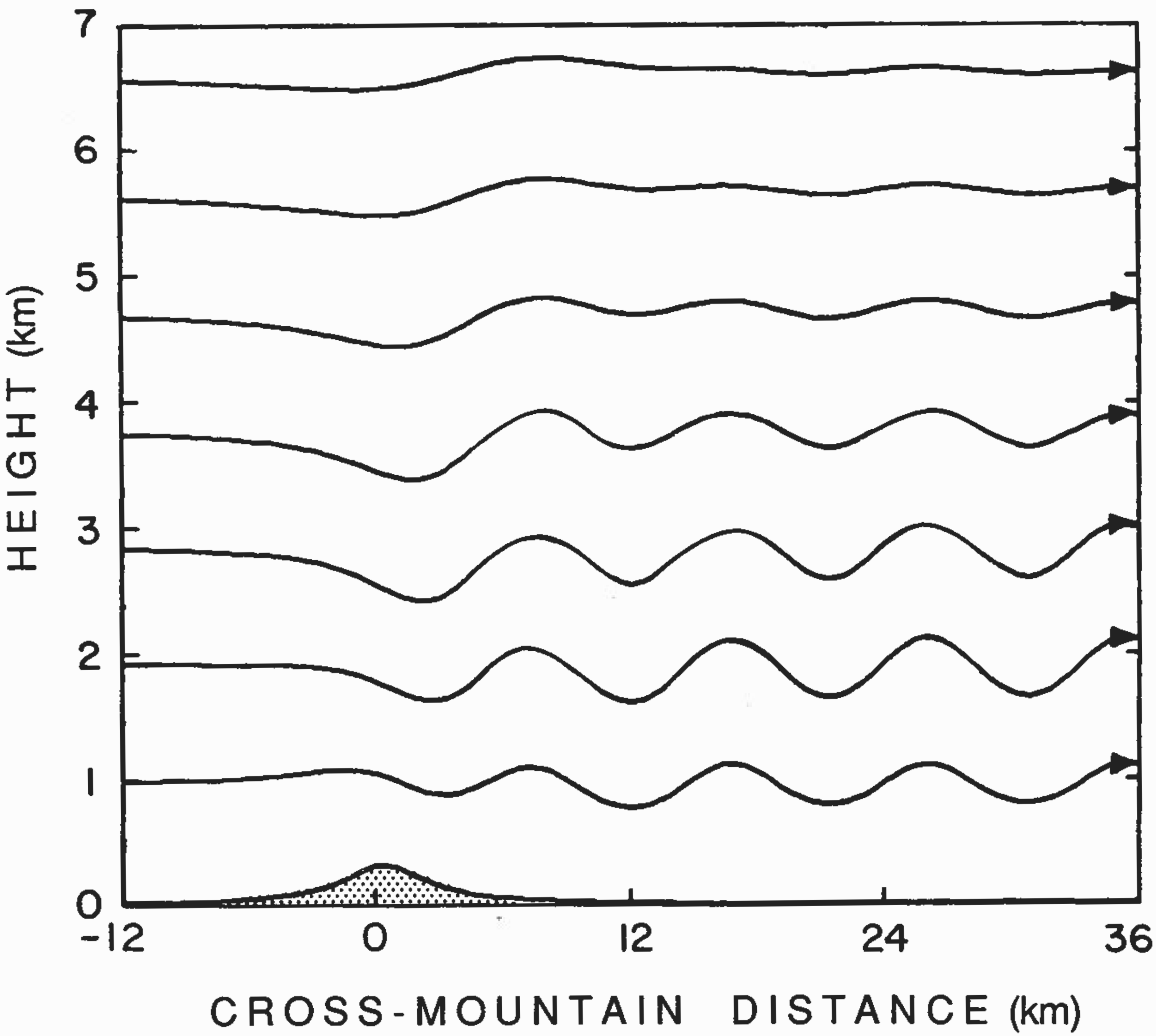


FIG. 4.4. Streamlines in steady airflow over an isolated ridge when the vertical variation in the Scorer parameter permits trapped waves.

(e.g. weaker stratification)

(e.g. stronger stratification)

Fig 2 Trapped lee waves

Durran, AMS, 1990

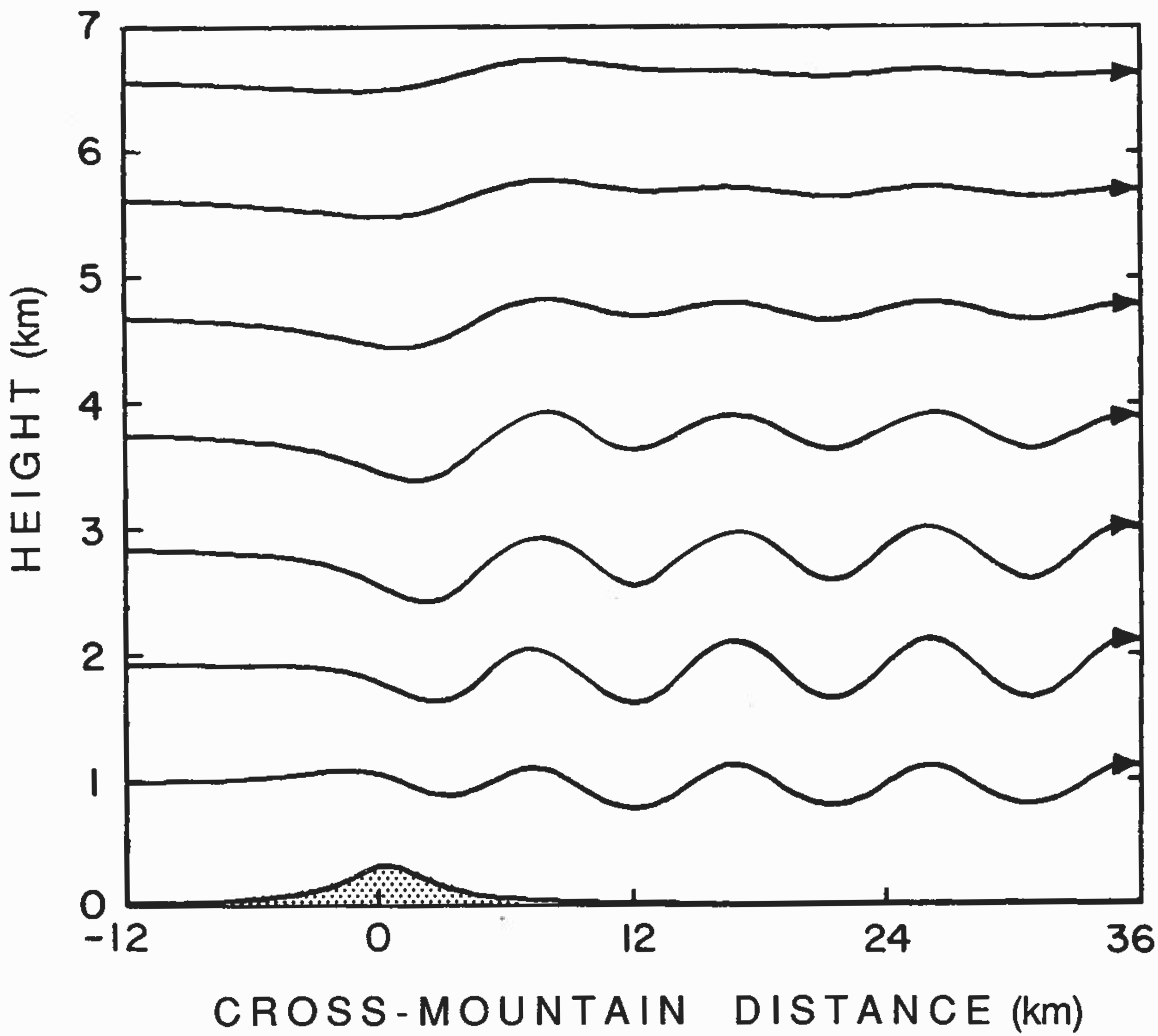


FIG. 4.4. Streamlines in steady airflow over an isolated ridge when the vertical variation in the Scorer parameter permits trapped waves.

(e.g. weaker stratification)

No phase tilt with height as not propagating upwards in net

(e.g. stronger stratification)

Fig 2 Trapped lee waves

Durran, AMS, 1990

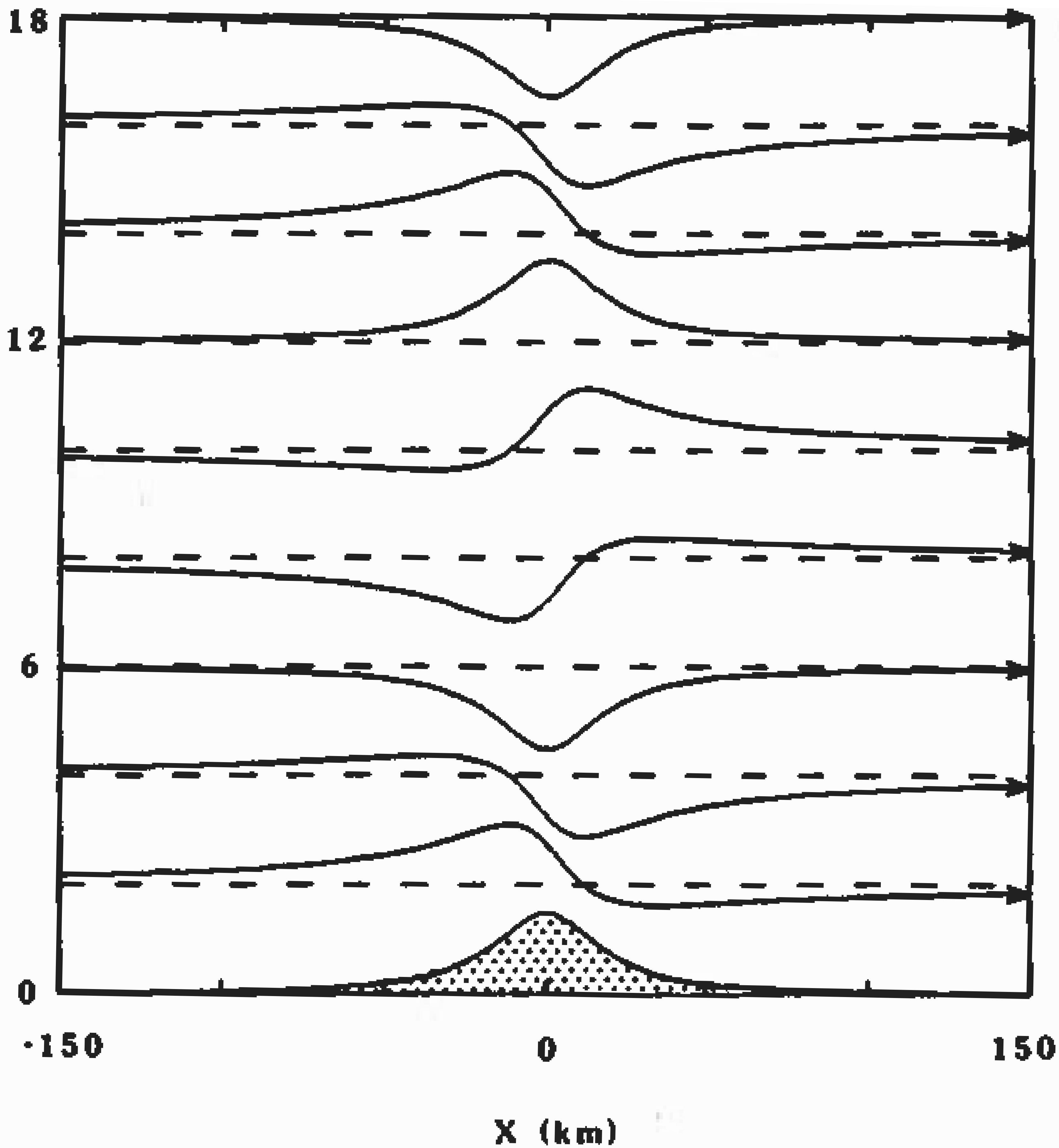
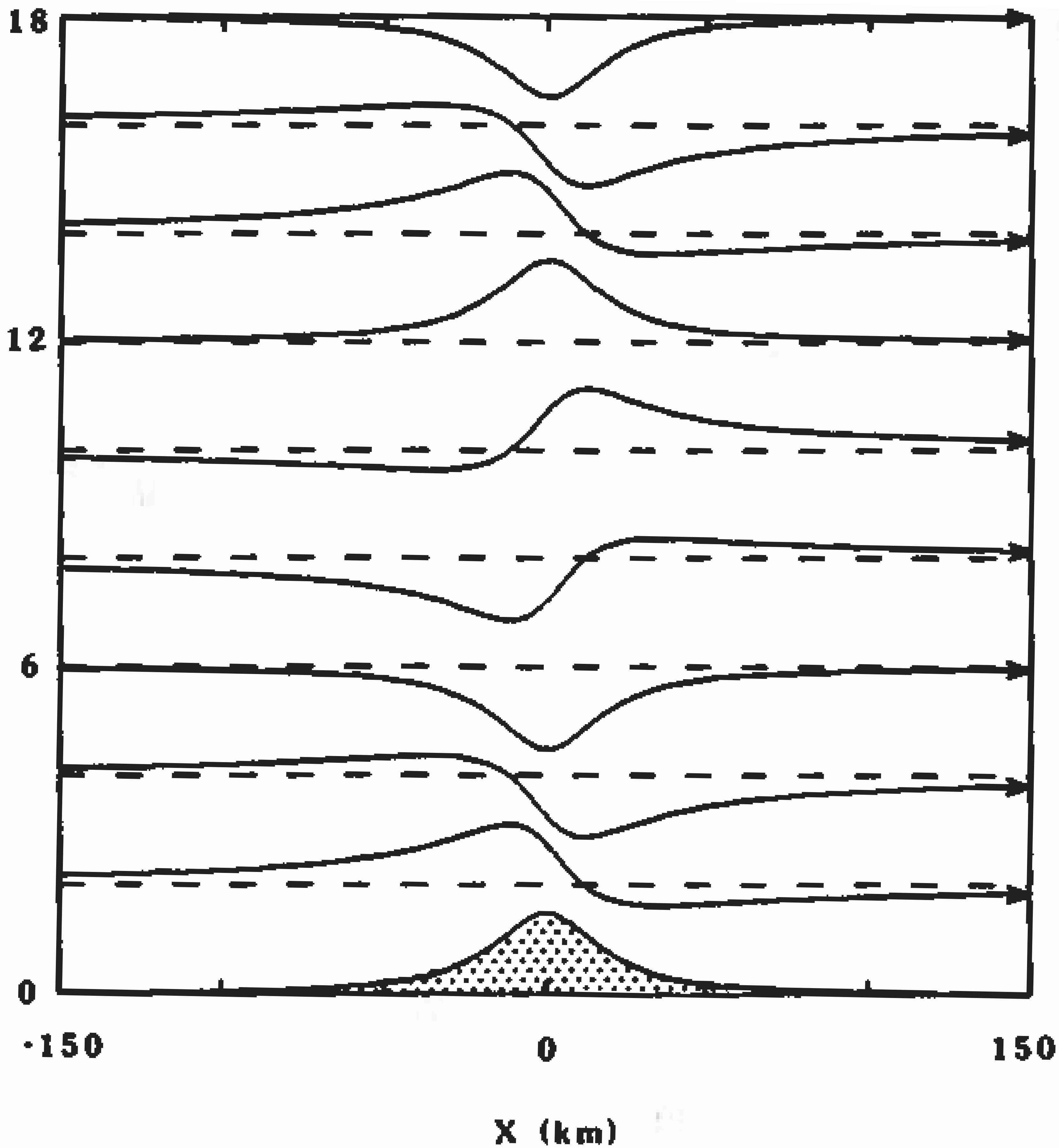


Fig 3 Waves over a *broad* isolated ridge

Durran, AMS, 1990



Because broad ridge, flow is periodic in the vertical (where does the ridge repeat itself?)

Fig 3 Waves over a *broad* isolated ridge

Durran, AMS, 1990

Introduction to pressure coordinates

- If make the hydrostatic approximation, then often useful to change from z to p as the vertical coordinate

- Advantages:
  - No time derivative in mass continuity equation
  - No 1/density in horizontal pressure force term

- Disadvantages:
  - Lower boundary pressure is not fixed in time
  - Static stability parameter not roughly constant in the vertical

- Note that:
  - Unit vector in vertical remains in the same direction
  - Derivatives in x and y change because now holding p rather than z constant

Introduction to pressure coordinates (handout)

Vertical velocity

The vertical velocity in pressure coordinates is given by $\omega = Dp/Dt$. This is analogous to how we define the horizontal velocities ($u = Dx/Dt$ and $v = Dy/Dt$) or the vertical velocity in height coordinates ($w = Dz/Dt$).

Lagrangian derivative

The Lagrangian derivative is expressed in pressure coordinates as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \omega \frac{\partial}{\partial p},$$

where $\mathbf{u} = (u, v)$. Both the horizontal gradient ∇ and the time derivative $(\partial/\partial t)$ are taken at constant p rather than z.

Introduction to pressure coordinates (handout)

Mass continuity equation

Mass conservation for a material element of air may be written as

$$\frac{D\rho\delta V}{Dt} = 0,$$

where $\delta V = \delta x \delta y \delta z$ is the volume of the material element. Hydrostatic balance gives us that $\rho \delta z = -\delta p / g$, such that

$$\frac{D\delta x \delta y \delta p}{Dt} = 0.$$

We then use that $D\delta x / Dt = \delta u$ where δu is the change in u across the material element in the x direction. Similarly $D\delta y / Dt = \delta v$ and $D\delta p / Dt = \delta \omega$. Substituting gives that

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0.$$

In the limit of an infinitesimal parcel of air, this then gives the mass continuity equation in pressure coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0.$$

Introduction to pressure coordinates (handout)

Pressure force in the horizontal

The pressure force term in the horizontal momentum equation in z coordinates may be written as

$$-\frac{1}{\rho}\left(\nabla p\right)_z,$$

where the subscript z make explicit that horizontal derivatives are taken at constant z. To convert this to pressure coordinates, we first write the general rule for converting a derivative with respect to x from the z vertical coordinate to the p vertical coordinate:

$$\left(\frac{\partial}{\partial x}\right)_p = \left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial}{\partial z}.$$

Applying this rule to the derivative of p gives that

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}.$$

Using hydrostatic balance then gives that

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \phi}{\partial x}\right)_p,$$

where $\phi = gz$ is the geopotential. Finally, considering derivatives with respect to both x and y gives that

$$-\frac{1}{\rho}\left(\nabla p\right)_z = -\left(\nabla \phi\right)_p.$$

Introduction to pressure coordinates (handout)

Hydrostatic balance (vertical momentum equation)

Hydrostatic balance in z coordinates ($\partial p/\partial z = -\rho g$) is more conveniently written using the ideal gas law as

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p}.$$

Thermodynamic equation

The thermodynamic equation in the absence of diabatic heating and written in terms of potential temperature (θ) remains:

$$\frac{D\theta}{Dt} = 0.$$

Introduction to pressure coordinates (handout)

Summary

The equations for horizontal velocity, hydrostatic balance, mass continuity and potential temperature in pressure coordinates in the absence of friction, diabatic heating, and planetary rotation are:

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} &= -\nabla\phi, \\
\frac{\partial\phi}{\partial p} &= -\frac{RT}{p}, \\
\nabla\cdot\mathbf{u} + \frac{\partial\omega}{\partial p} &= 0, \\
\frac{D\theta}{Dt} &= 0.
\end{aligned}
$$

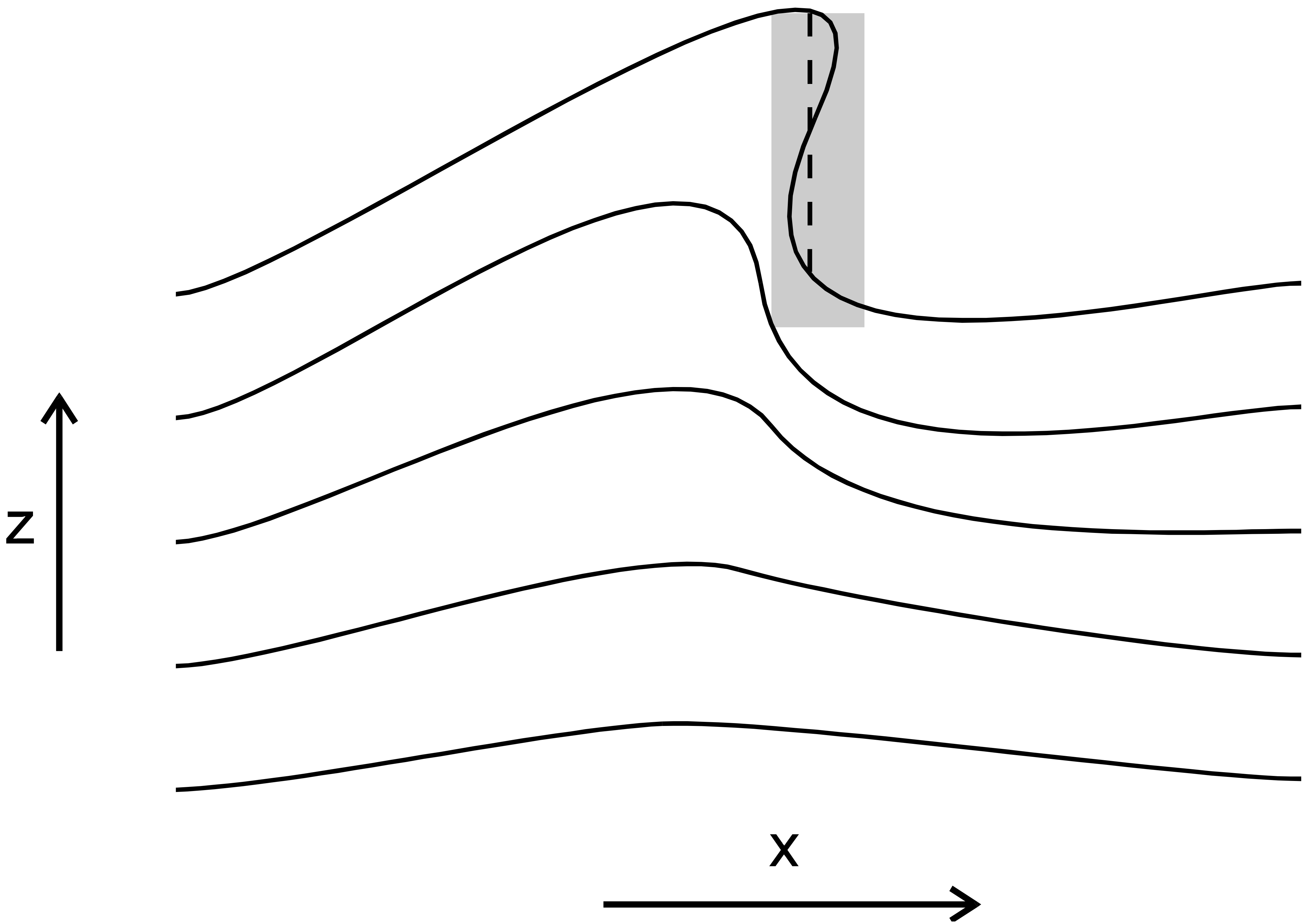


Fig 4a Contours of potential temperature in a breaking gravity wave

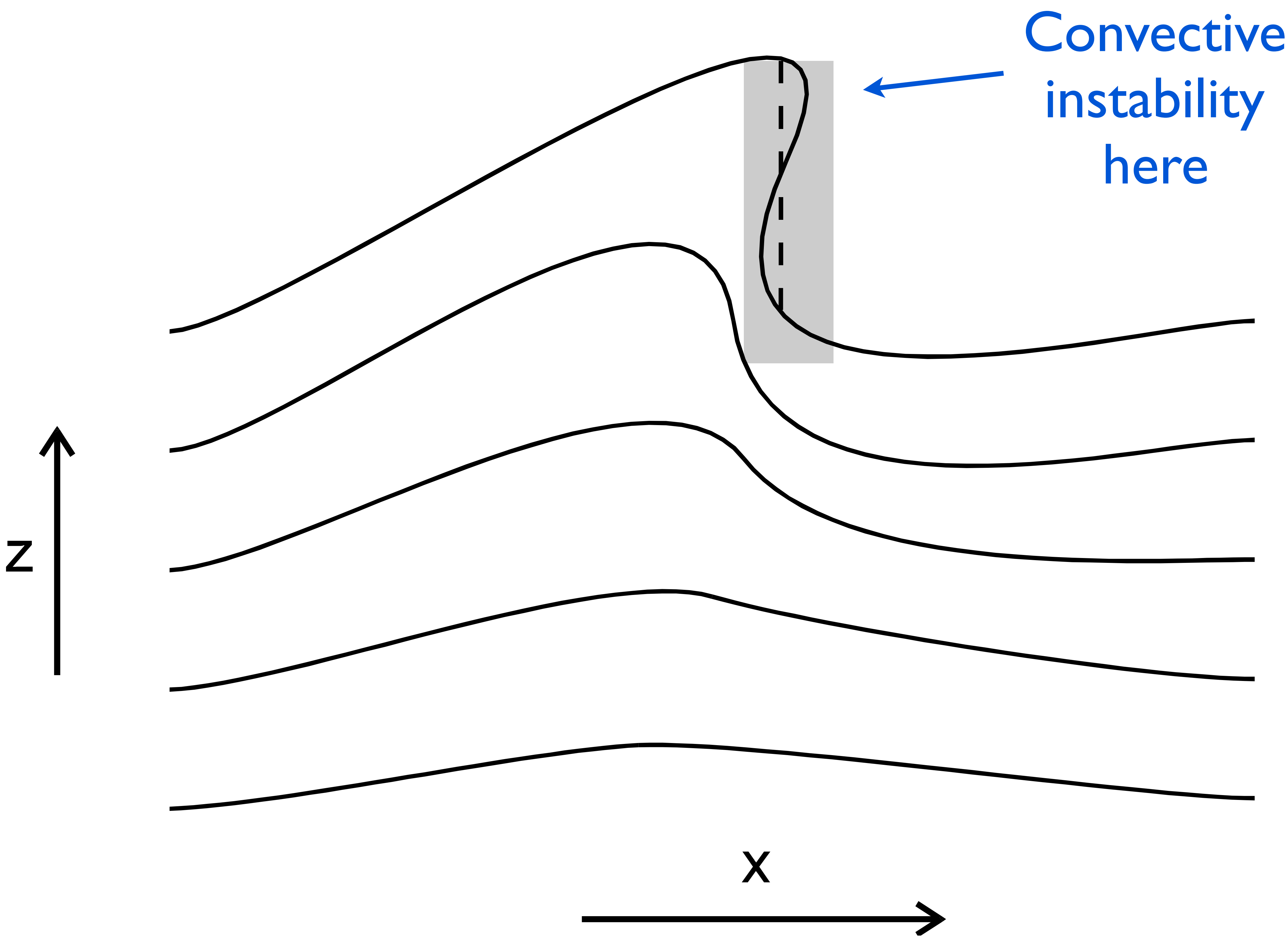


Fig 4a Contours of potential temperature in a breaking gravity wave

Non-acceleration theorem for stationary gravity waves (handout)

We have seen that for a 2-D flow (in x and z) the mean state is affected by waves through convergence of vertical fluxes of temperature ($\rho\overline{w'\theta'}$) and momentum ($\rho\overline{w'u'}$) by the waves. The overline denotes a zonal mean and the primes denote wave quantities. In this handout, we will derive expressions for how these fluxes vary in the vertical following Eliassen and Palm 1961.

We assume that the waves are stationary, inviscid, adiabatic and small amplitude. The assumption of small-amplitude waves allows us to use the linearized equations of motion to calculate the wave fluxes. The wave fluxes will change the mean state, but the changes in mean state are considered as a higher-order correction in our calculation of the fluxes.

Non-acceleration theorem for stationary gravity waves (handout)

Consider a basic state defined by zonal wind $U_0(z)$ and potential temperature $\theta_0(z)$. We assume the basic state is statically stable such that $\frac{\partial \theta_0}{\partial z} > 0$. Given the assumption of stationary waves $(\partial/\partial t = 0)$, the linearized equations in log-pressure coordinates are:

$$U_0 \frac{\partial u'}{\partial x} + w' \frac{\partial U_0}{\partial z} + \frac{\partial \phi'}{\partial x} = 0, \tag{5}$$

$$\frac{\partial u'}{\partial x} + \frac{1}{\rho} \frac{\partial \rho w'}{\partial z} = 0, \tag{6}$$

$$U_0 \frac{\partial \theta'}{\partial x} + w' \frac{\partial \theta_0}{\partial z} = 0, \tag{7}$$

$$\frac{\partial \phi'}{\partial z} - \frac{R \Pi \theta'}{H} = 0. \tag{8}$$

Non-acceleration theorem for stationary gravity waves (handout)

First consider the vertical wave flux of temperature ($\rho\overline{w'\theta'}$). Multiple Eq. 7 by θ' and take the zonal average (i.e., average in x):

$$
\begin{aligned}
U_0\,\overline{\theta'\frac{\partial\theta'}{\partial x}} + \overline{w'\theta'}\frac{\partial\theta_0}{\partial z} &= 0,\\
\Rightarrow U_0\,\overline{\frac{1}{2}\frac{\partial\theta'^2}{\partial x}} + \overline{w'\theta'}\frac{\partial\theta_0}{\partial z} &= 0,\\
\Rightarrow \overline{w'\theta'}\frac{\partial\theta_0}{\partial z} &= 0,
\end{aligned}
$$

where the last step follows because we assume the waves are periodic in x. Thus, we have zero vertical temperature flux ($\rho\overline{w'\theta'} = 0$) by our stationary and adiabatic waves.

Non-acceleration theorem for stationary gravity waves (handout)

Next we consider the vertical momentum flux ($\rho\overline{w'u'}$). This flux is generally not zero, but how does it vary in the vertical?

Multiply Eq. 5 by u' and average in x to give an equation that is the budget of kinetic energy of the waves (if we had not assumed stationary waves there would be a term $\partial({u'}^2/2)/\partial t$):

$$
\begin{aligned}
U_0\overline{\frac{\partial u'}{\partial x}u'} + \overline{w'u'}\frac{\partial U_0}{\partial z} + \overline{u'\frac{\partial \phi'}{\partial x}} &= 0 \\
\Rightarrow \overline{w'u'}\frac{\partial U_0}{\partial z} + \overline{u'\frac{\partial \phi'}{\partial x}} &= 0.
\end{aligned}
$$

Thus, the pressure force term (in the form of $\frac{\partial \phi'}{\partial x}$) in the zonal momentum equation causes $\overline{w'u'} \neq 0$ unlike for $\overline{w'\theta'}$.

Non-acceleration theorem for stationary gravity waves (handout)

Several steps later…

The final form of our energy equation is

$$\boxed{\rho\overline{w'u'}\frac{\partial U_0}{\partial z} + \frac{\partial \rho\overline{w'\phi'}}{\partial z} = 0,} \tag{9}$$

where the first term represents conversion between wave and mean energy, and the second term is the divergence of the vertical wave energy flux. We have found a relation between the vertical momentum flux $\rho\overline{w'u'}$ and the vertical wave energy flux $\rho\overline{w'\phi'}$, but we will need another constraint to find either flux individually.

Non-acceleration theorem for stationary gravity waves (handout)

We go back to the wave zonal momentum equation (Eq. 5) and group the x-derivatives together:

$$\frac{\partial}{\partial x}(U_0 u' + \phi') + w' \frac{\partial U_0}{\partial z} = 0.$$

We next multiply by $U_0 u' + \phi'$ (last time we multiplied by u') to give

$$\frac{\partial}{\partial x}\left[\frac{1}{2}(U_0 u' + \phi')^2\right] + U_0 u' w' \frac{\partial U_0}{\partial z} + w' \phi' \frac{\partial U_0}{\partial z} = 0.$$

Several steps later…

$$\boxed{U_0 \rho \overline{u'w'} + \rho \overline{w'\phi'} = 0.} \qquad (10)$$

Non-acceleration theorem for stationary gravity waves (handout)

Substituting for $\rho\overline{w'\phi'}$ from Eq. 10 into Eq. 9 gives that:

$$\rho\overline{w'u'}\frac{\partial U_0}{\partial z} - \frac{\partial}{\partial z}(U_0\rho\overline{u'w'}) = 0$$

$$\Rightarrow \rho\overline{w'u'}\frac{\partial U_0}{\partial z} - \frac{\partial U_0}{\partial z}\rho\overline{w'u'} - U_0\frac{\partial}{\partial z}(\rho\overline{u'w'}) = 0.$$

Our final and simple result is that

$$\boxed{U_0\frac{\partial}{\partial z}(\rho\overline{u'w'}) = 0.} \qquad (11)$$

The vertical wave momentum flux $\rho\overline{u'w'}$ is constant in the vertical $(\partial\rho\overline{u'w'}/\partial z = 0)$ except where $U_0 = 0$. Since we have previously shown that

$$\frac{\partial\overline{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial z}(\rho\overline{w'u'}),$$

we conclude that small-amplitude, adiabatic, inviscid, and stationary waves do not affect the mean flow except where $U_0 = 0$.

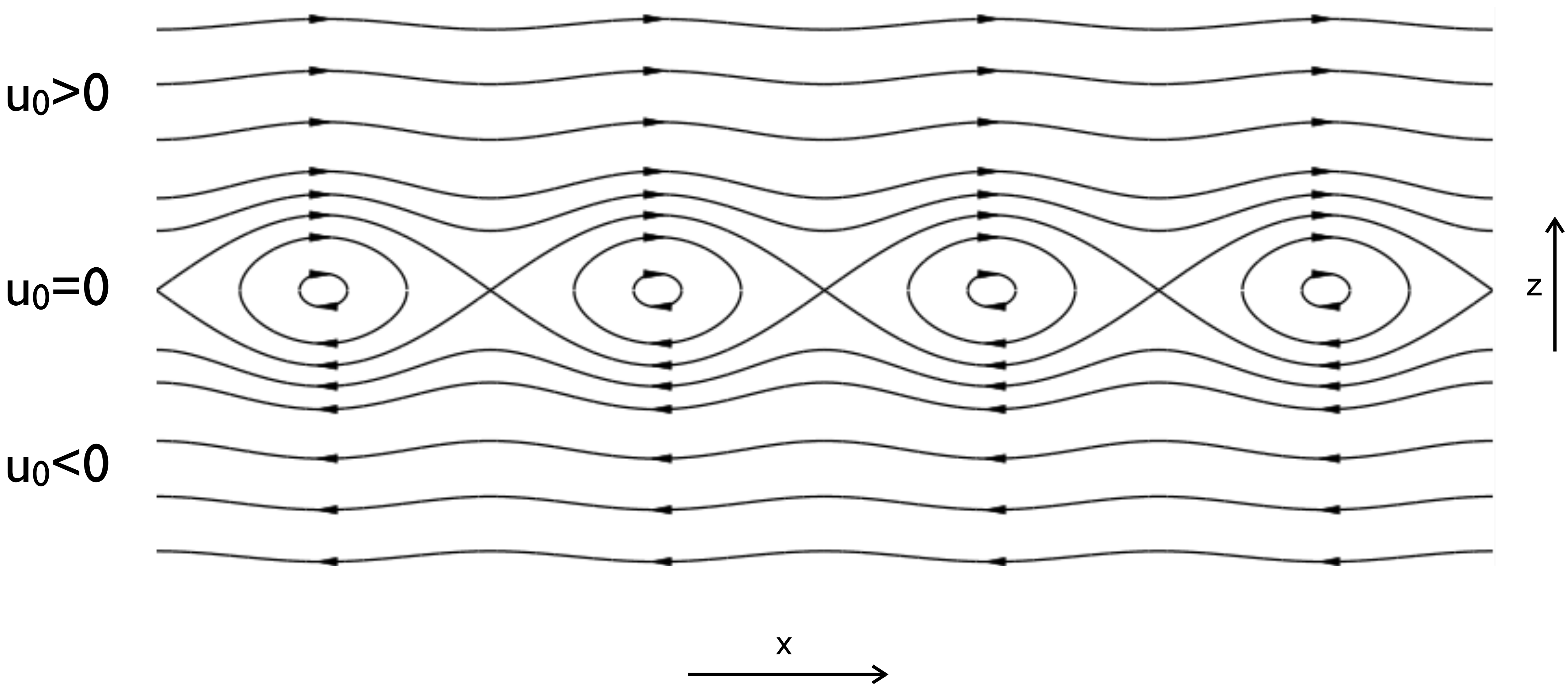


Fig 4b   "Cat's eye" flow at a critical level
Contours are the streamfunction in a frame with c=0

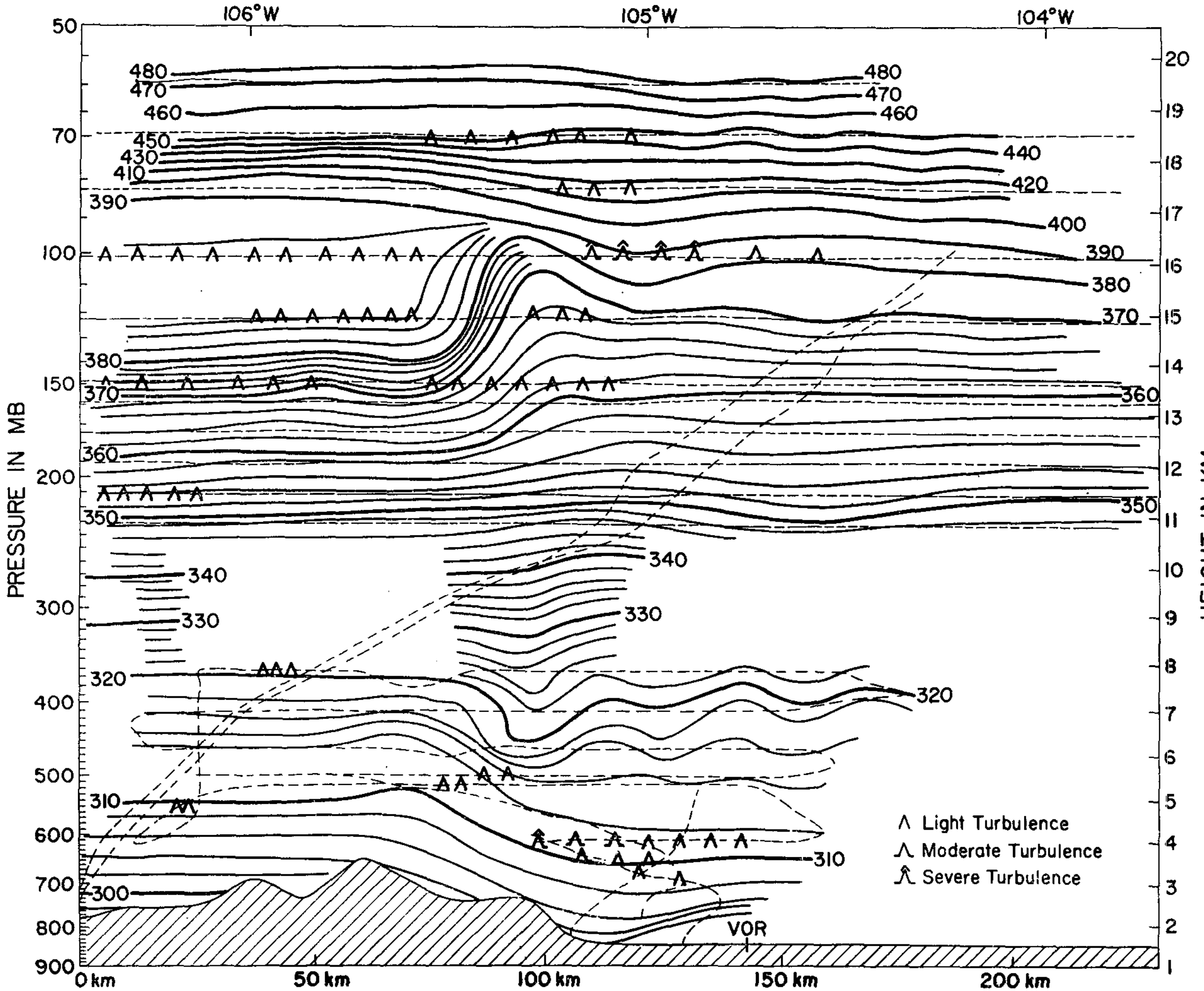


FIG. 1. Potential temperature cross section for 17 February 1970. Solid lines are isentropes (°K), dashed lines aircraft or balloon flight trajectories. The cross section is along a 275°–095° true azimuth line, crossing the Kremmling, Colo., and Denver VOR aircraft navigation stations.

Fig 5 Example of mountain wave over Rockies *Lilly et al, JAS, 1973*

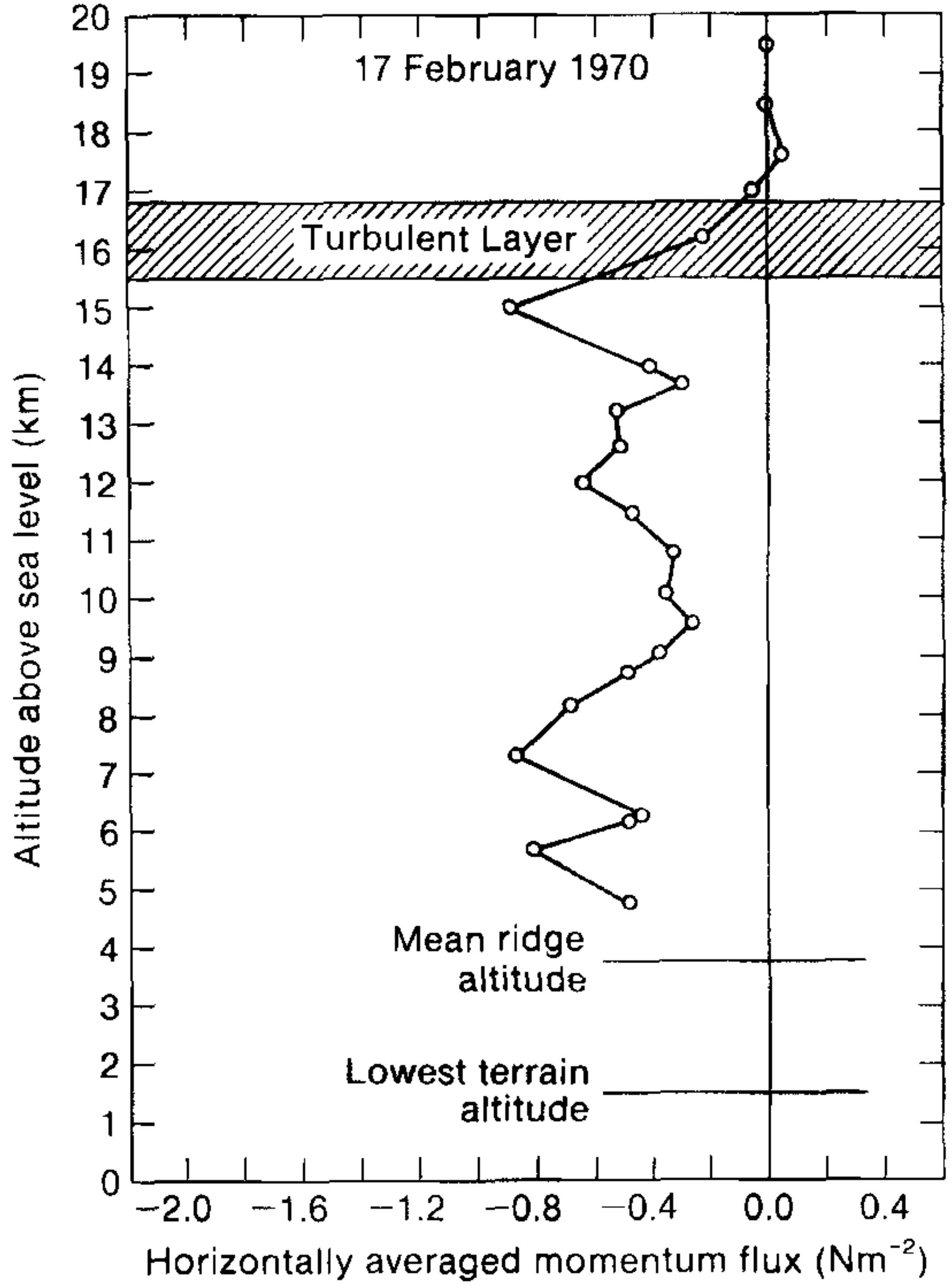


Figure 3. Mean observed profile of momentum flux over the Rocky mountains on 17 February 1970 (after Lilly and Kennedy 1973).

Fig 6 Observed vertical momentum flux in mountain wave over Rockies

Palmer et al, QJRMS, 1986

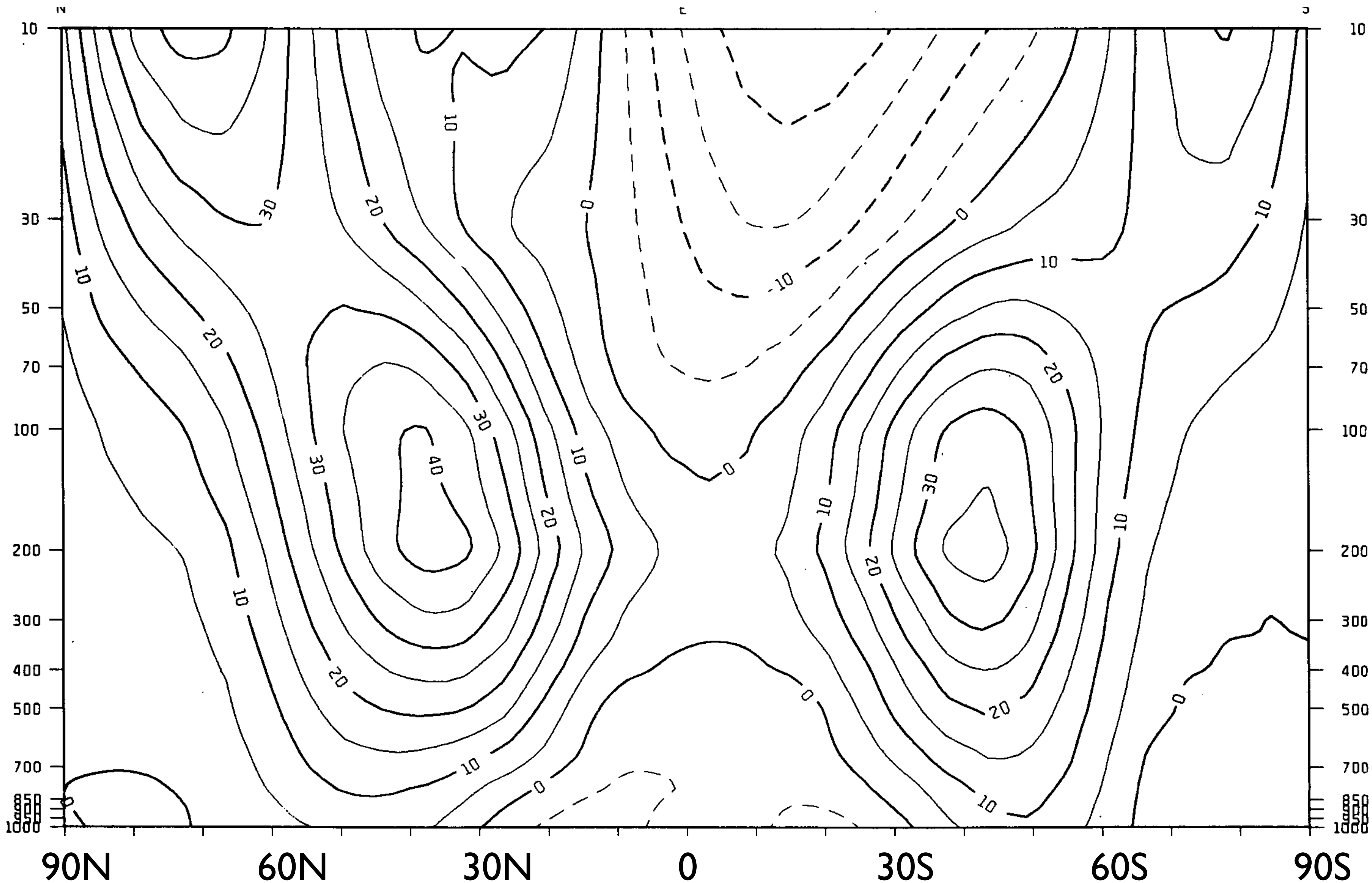


Fig 7 Zonal-mean zonal winds (m/s) in control simulation of general circulation model (GCM) *without* gravity-wave drag parameterization (season is DJF)

McFarlane, JAS, 1987

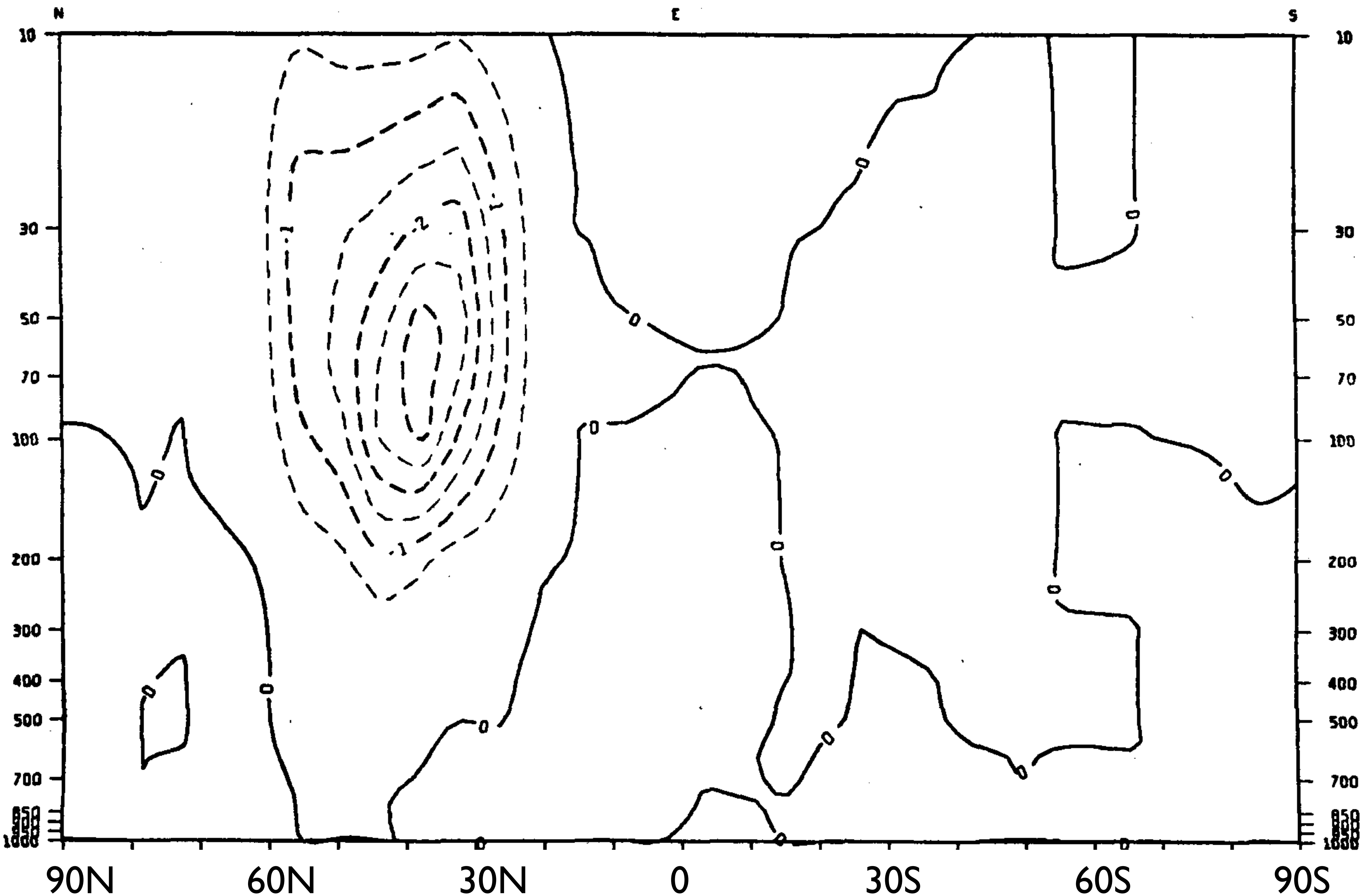


Fig 8

Deceleration of zonal-mean zonal wind by orographic gravity-wave drag parameterization

McFarlane, JAS, 1987
Units: m/s/day

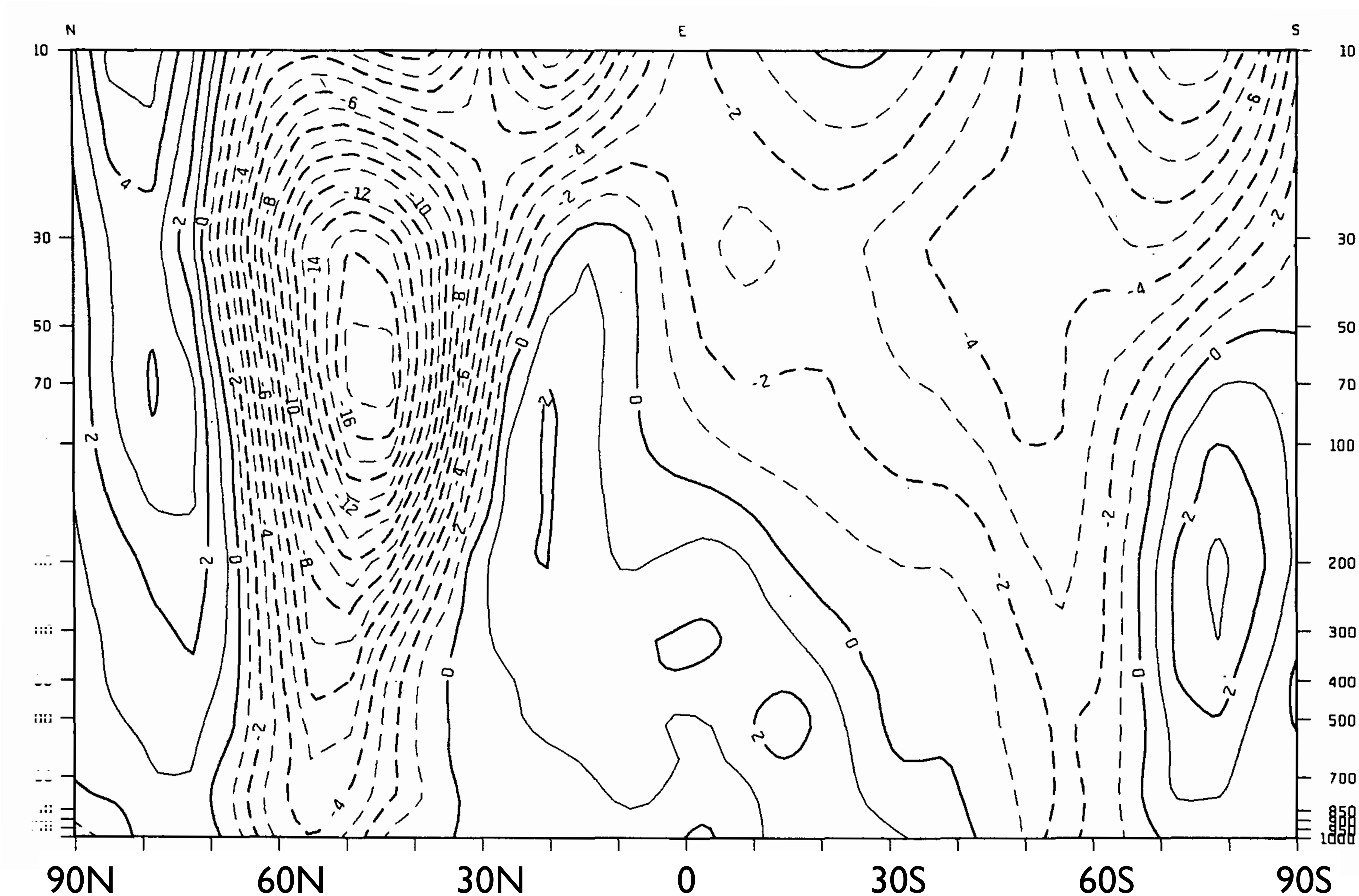


Fig 9

Change in zonal-mean zonal wind (m/s)
(simulation with gravity-wave drag minus control)

McFarlane, JAS, 1987

Slowing of westerlies

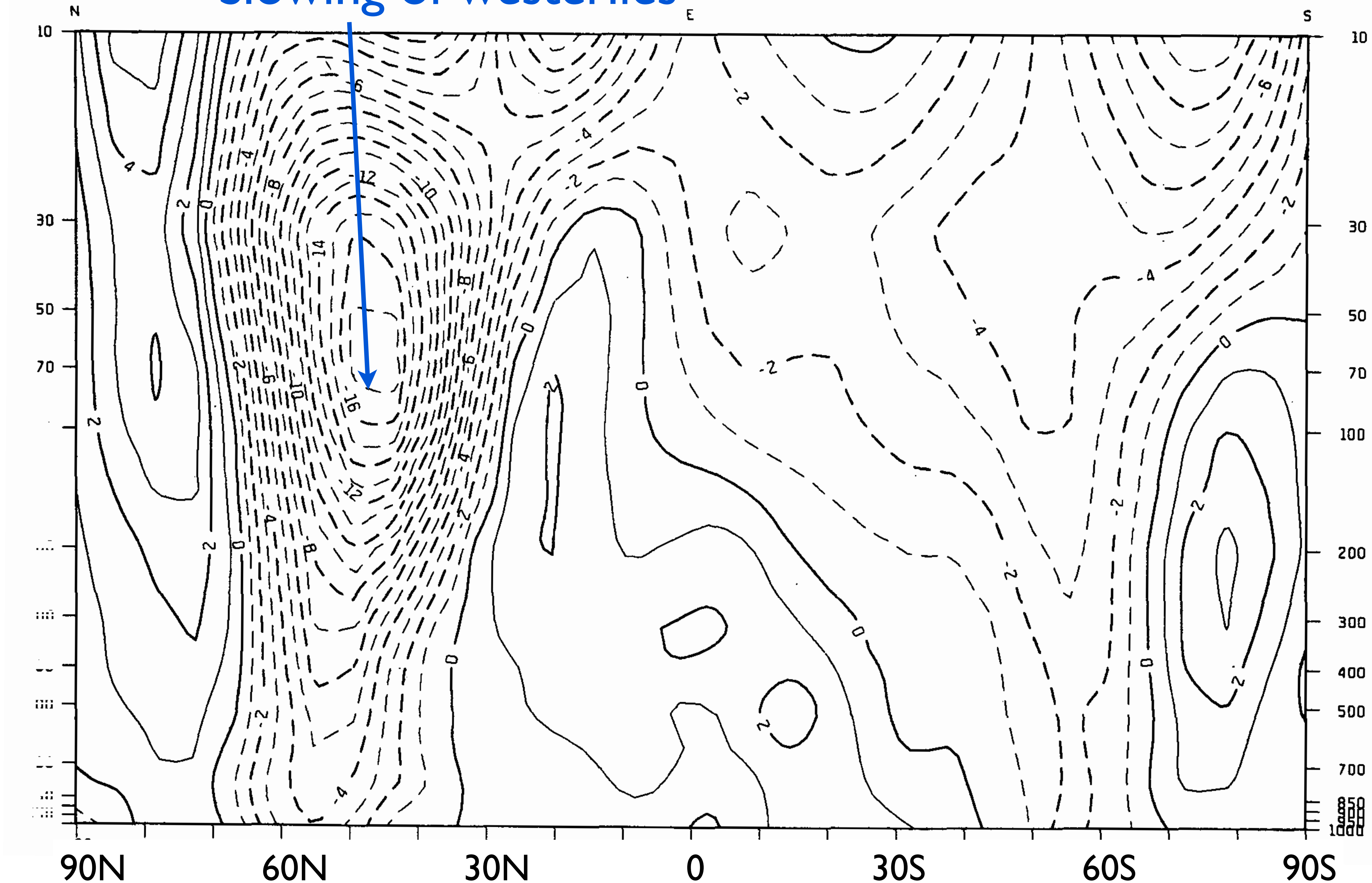
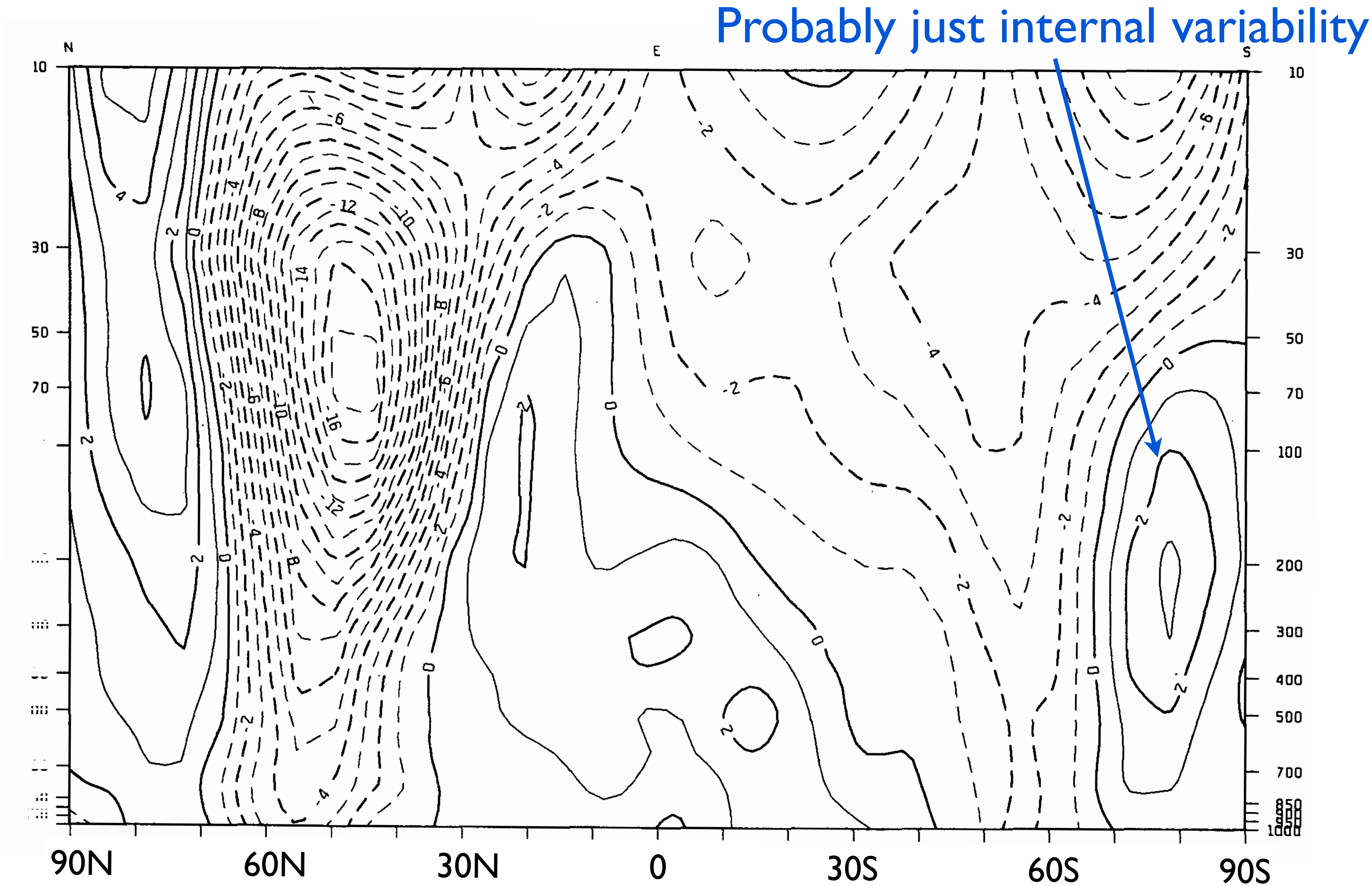


Fig 9

Change in zonal-mean zonal wind (m/s)
(simulation with gravity-wave drag minus control)

McFarlane, JAS, 1987



Change in zonal-mean zonal wind (m/s)
(simulation with gravity-wave drag minus control)

Fig 9

McFarlane, JAS, 1987

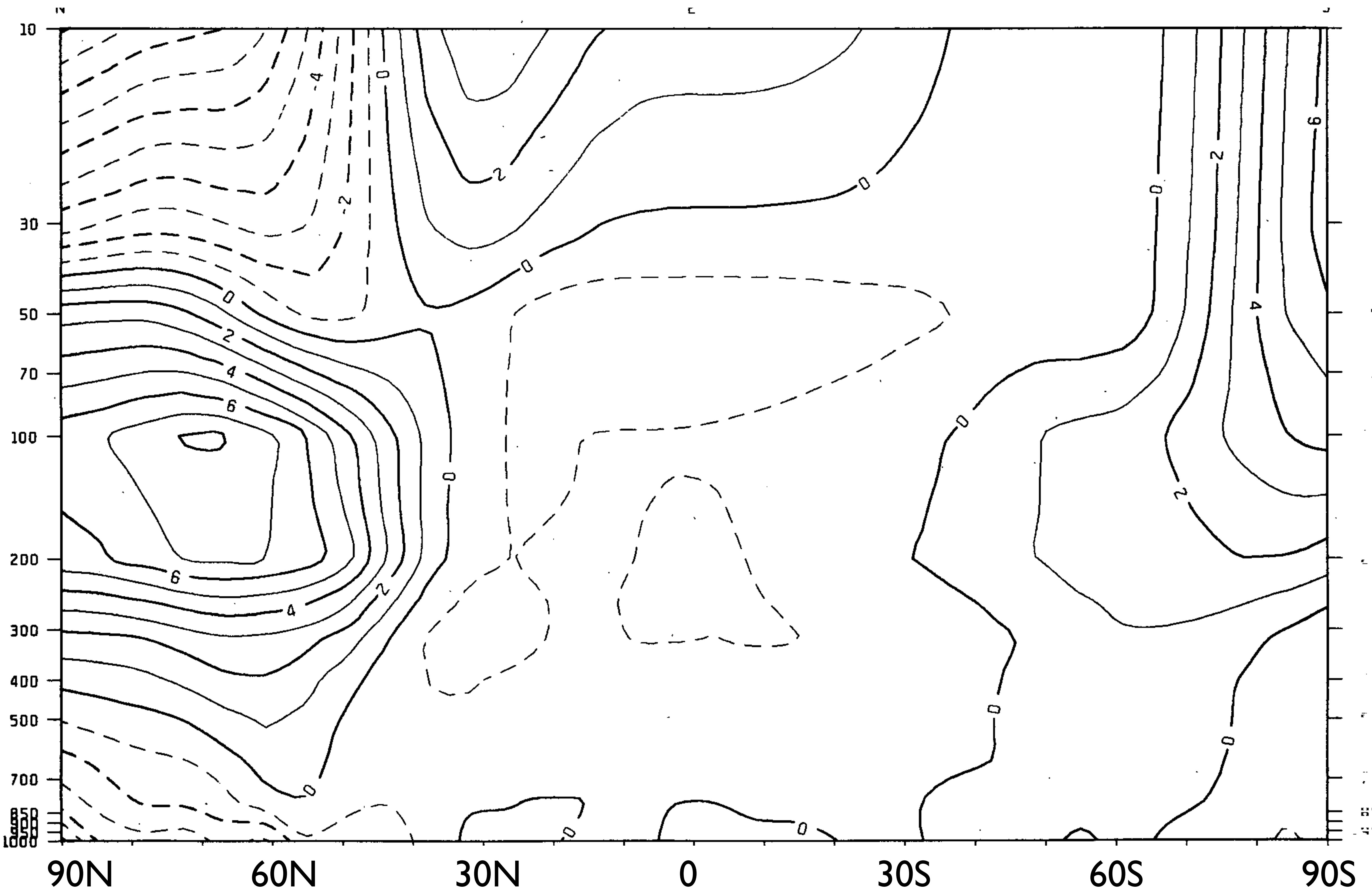


Fig 10

Change in zonal-mean temperature (K)
(simulation with gravity wave drag minus control)

McFarlane, JAS, 1987

Warming to keep thermal wind balance

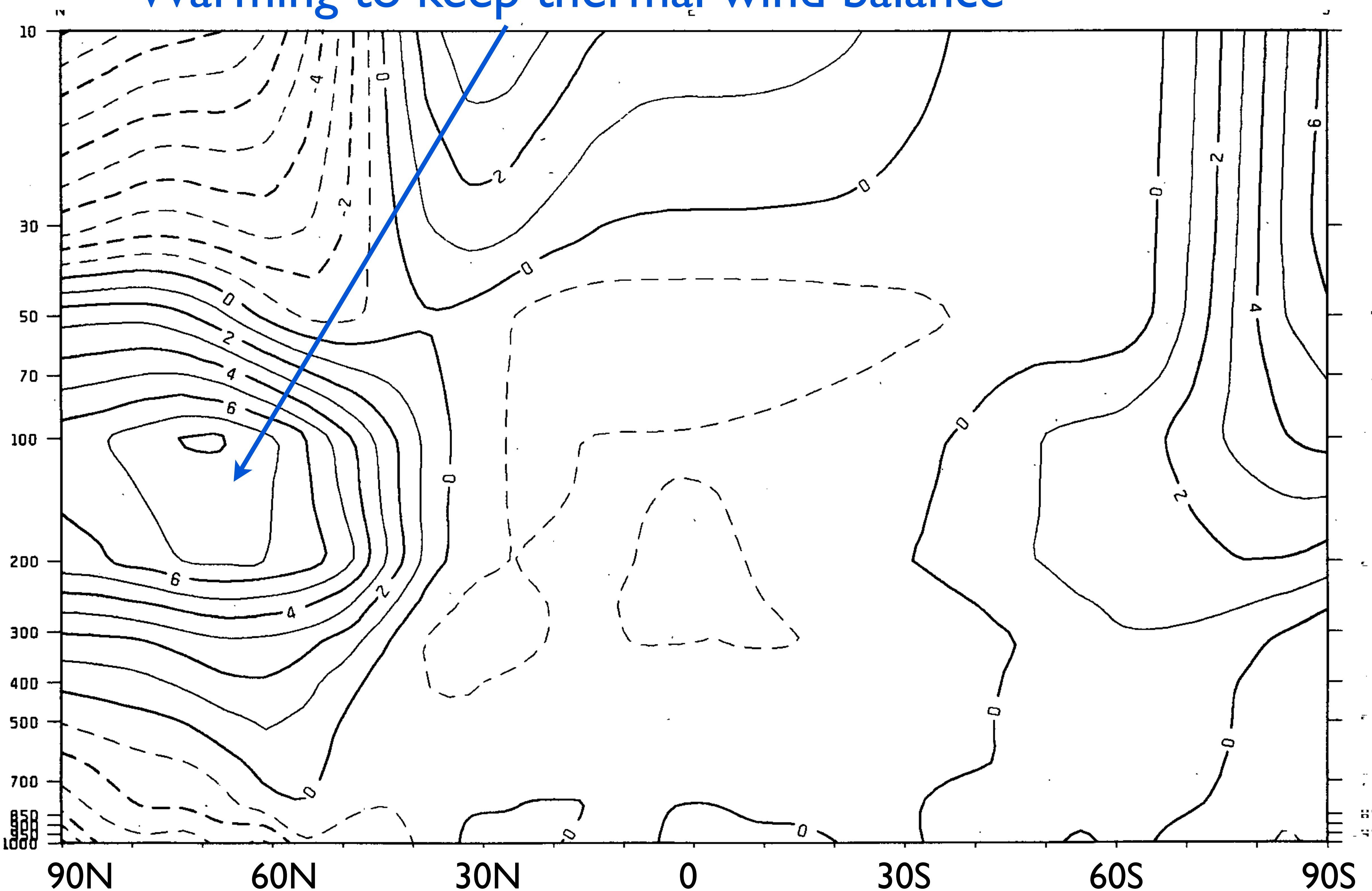


Fig 10

Change in zonal-mean temperature (K)
(simulation with gravity wave drag minus control)

McFarlane, JAS, 1987

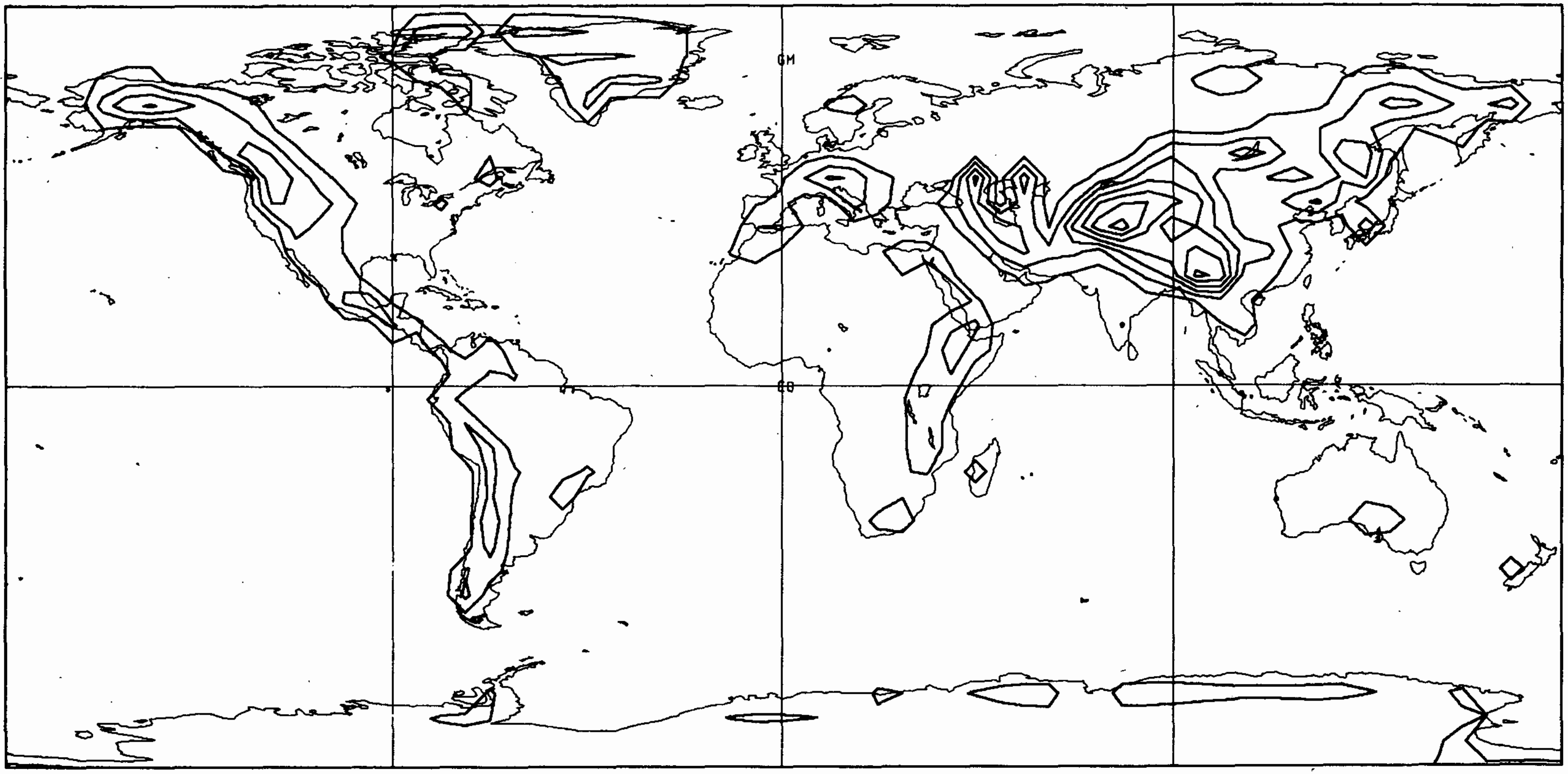


FIG. 21. Net stress drop over the vertical model domain due to the wave drag force. Contours are for 0.05 Pa and larger with an interval of 0.1 Pa.

Fig 11 Magnitude of orographic gravity wave drag stress on the atmosphere (Pa)

McFarlane, JAS, 1987

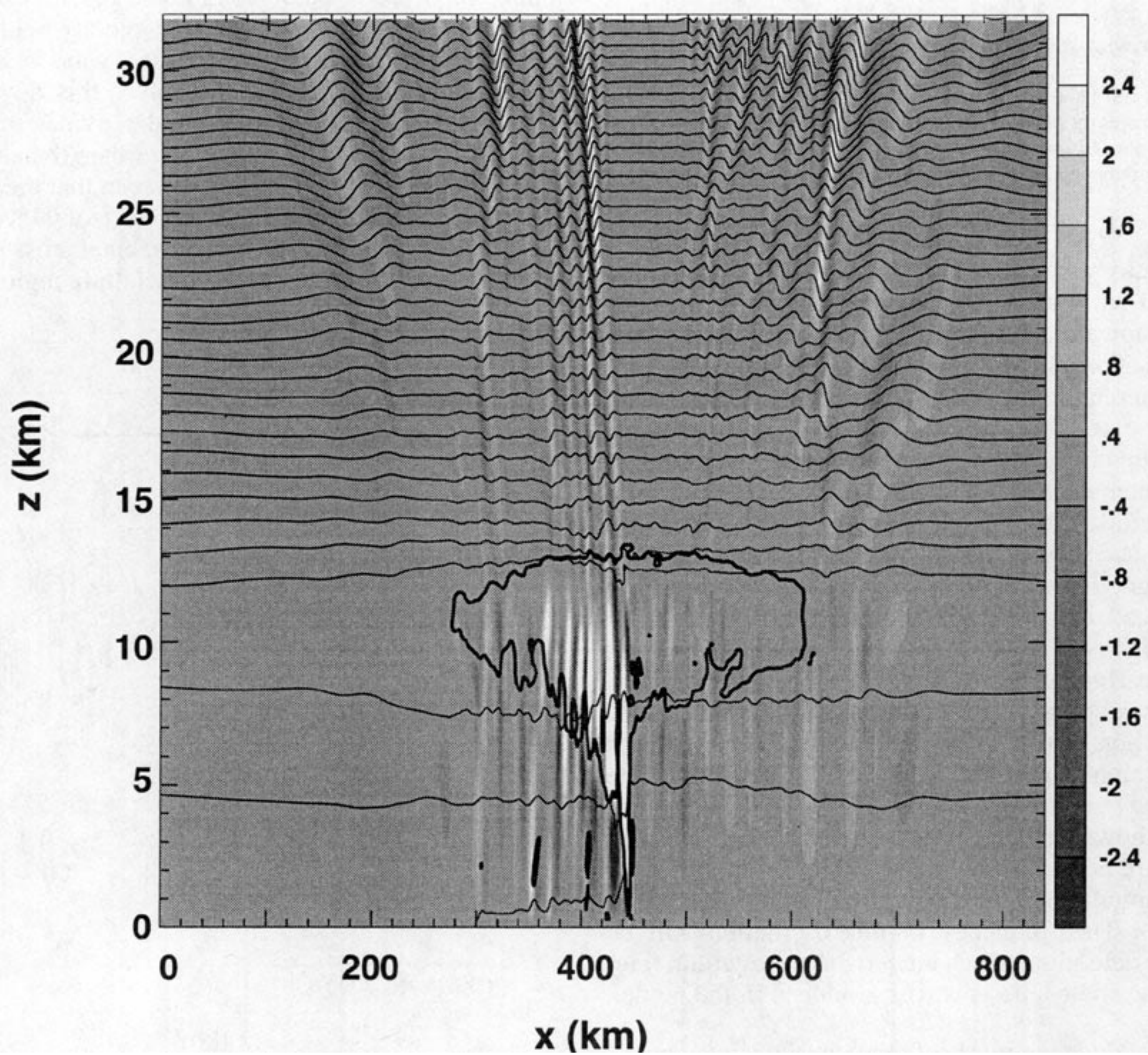


Fig 12a

Gravity waves generated by a squall line and propagating vertically and horizontally

FIG. 2. The squall line simulation at 4 h of simulation time. Shading represents contours of vertical velocity. (Contrast has been enhanced to show the qualitative structure; the full range of vertical velocities is +20 to −5 m s^{-1}.) Thin lines are isentropes (at 15-K intervals), and the thick line shows the cloud outline (cloud water mixing ratio = 1×10^{-4} g g^{-1}). The tropopause is at 12–13 km.

Alexander et al, JAS, 1995

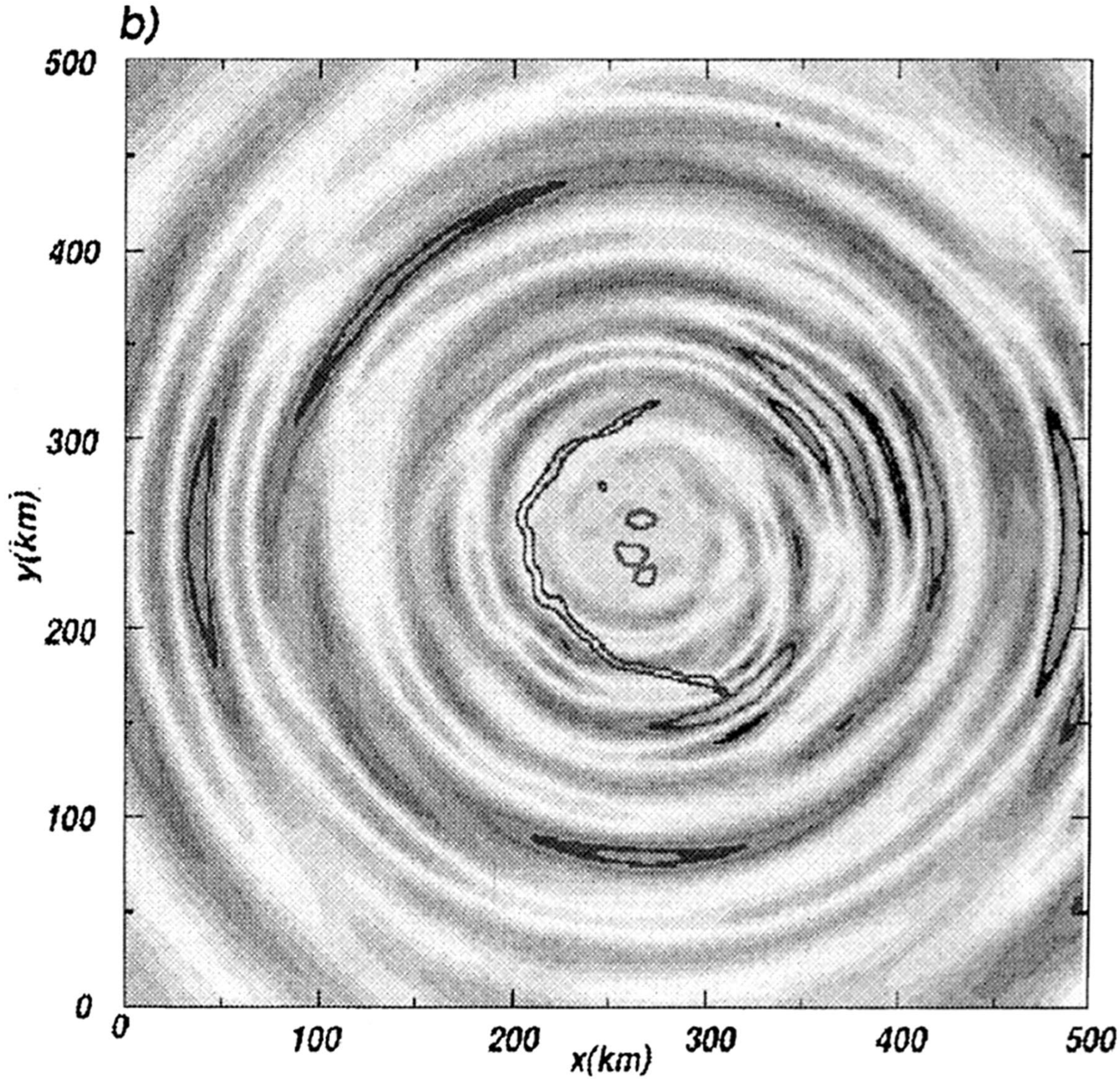


Fig 12b

Gravity waves at z=40km

Three-dimensional study of gravity waves generated by convection in a mesoscale model with parameterized microphysics.(b) The *x* -*y* cross section of vertical velocity at *z* 40 km. Also shown are the surface gust front (arc-shaped solid line) and regions of strong latent heating in the troposphere (small solid contours).

Fritts et al, Rev. Geophs. 2003

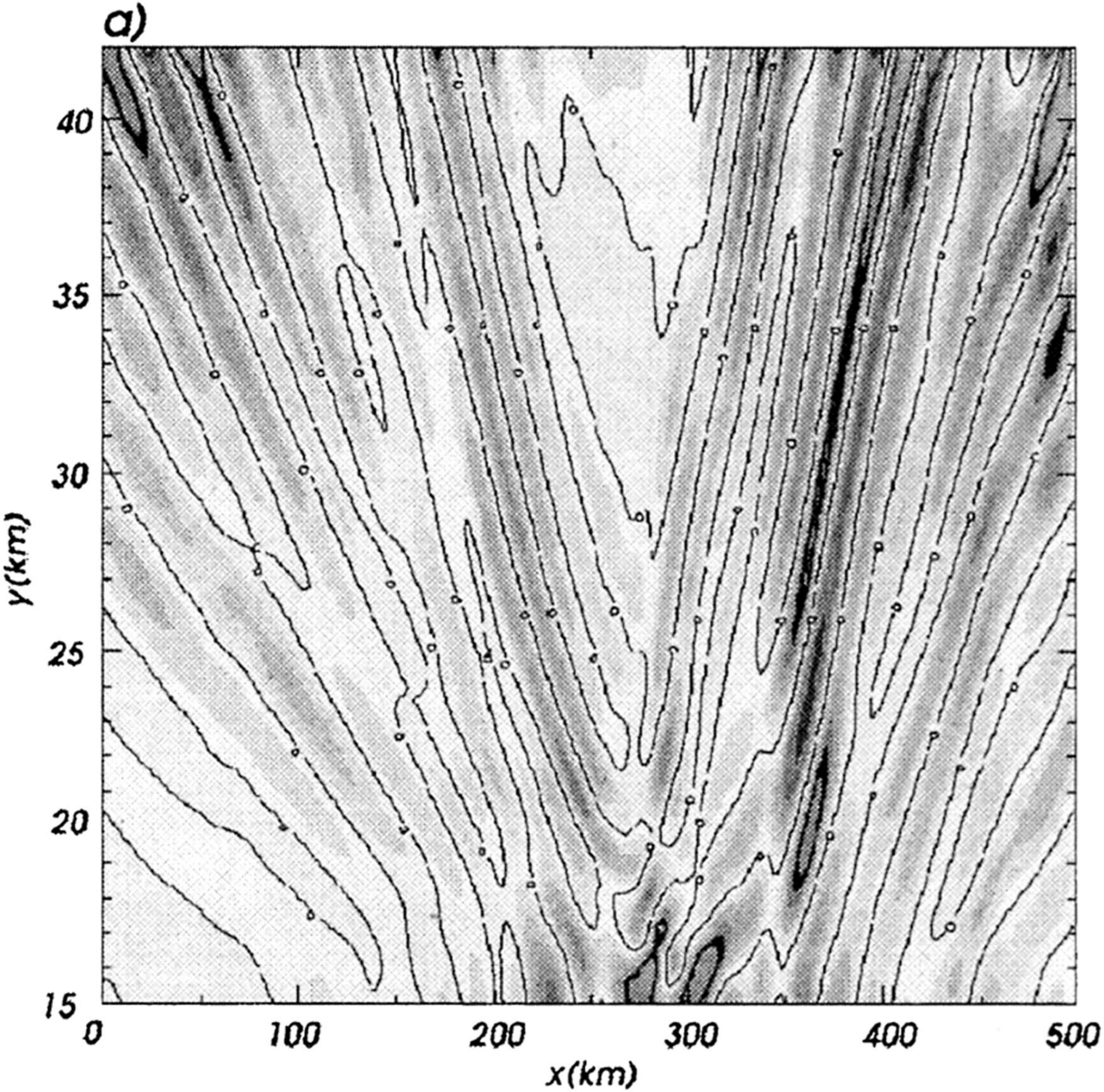


Fig 12b

Horizontal-vertical cross section

Three-dimensional study of gravity waves generated by convection in a mesoscale model with parameterized microphysics. (a) Vertical velocity patterns in a cross section in the vertical (z) and zonal (x) plane at y 250 km.

Fritts et al, Rev. Geophs. 2003