Introduction to Boussinesq and anelastic approximations for the atmosphere

We will make use of the Boussinesq approximation when discussing the Held-Hou model of the Hadley cell and at the start of the section on internal gravity waves. These notes present a brief introduction to the anelastic approximation which is commonly used in modeling of the atmosphere, followed by an additional approximation for shallow circulations that gives the Boussinesq equations. The Boussinesq approximation is not appropriate for deep circulations in the atmosphere such as the Hadley cell, but we will use it in the Held-Hou model to simplify the analysis.

Anelastic approximation

We define a reference state that is homentropic (constant potential temperature) and hydrostatically balanced. The reference state has pressure $p_0(z)$, density $\rho_0(z)$, and constant potential temperature θ_0, and it satisfies

$$\frac{\partial p_0}{\partial z} = -\rho_0 g,$$

where g is the acceleration due to gravity. The deviations from the reference state are assumed to be small. For simplicity of notation when using the final set of equations, we denote the total variables as p_{tot}, ρ_{tot}, and θ_{tot}, and the small perturbations as p, ρ and θ, such that

$$p_{\mathrm{tot}}(x,y,z,t) = p_0(z) + p(x,y,z,t), \tag{1}$$
$$\rho_{\mathrm{tot}}(x,y,z,t) = \rho_0(z) + \rho(x,y,z,t), \tag{2}$$
$$\theta_{\mathrm{tot}}(x,y,z,t) = \theta_0 + \theta(x,y,z,t), \tag{3}$$

The unapproximated equations omitting viscosity, diabatic heating and planetary rotation are:

$$\rho_{\mathrm{tot}} \frac{D\mathbf{u}}{Dt} = -\nabla p_{\mathrm{tot}} - \rho_{\mathrm{tot}} g \hat{\mathbf{z}}, \tag{4}$$
$$\frac{\partial \rho_{\mathrm{tot}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{tot}} \mathbf{u}) = 0, \tag{5}$$
$$\frac{D\theta}{Dt} = 0, \tag{6}$$

where the wind vector is $\mathbf{u} = (u, v, w)$, the Lagrangian derivative is $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$, and the unit vector in the vertical is $\hat{\mathbf{z}}$.

Substituting into the momentum equation (4) and neglecting the perturbation density on the left hand side gives

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla(p_0 + p) - (\rho_0 + \rho) g \hat{\mathbf{z}}.$$

The equations for the horizontal winds are then

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x},$$
$$\frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y},$$

where $\Phi = p/\rho_0$. More care is need with the vertical wind equation because we are interested in flows for which buoyancy is important and so we can't neglect the perturbation density when it is associated with gravity:

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial(p_0+p)}{\partial z} - \frac{g(\rho_0+\rho)}{\rho_0}, \quad (7)$$
$$= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0}, \quad (8)$$
$$= -\frac{\partial \Phi}{\partial z} - \frac{p}{\rho_0^2}\frac{\partial \rho_0}{\partial z} - \frac{g\rho}{\rho_0}. \quad (9)$$

We used that the reference state is hydrostatically balanced to go from (7) to (8). We next use potential temperature to simplify the right hand side of (9). The potential temperature of the reference state is $\theta_0 = T_0(p_r/p_0)^\kappa$ where p_r is a reference constant pressure (e.g., 1000hPa) not to be confused with the reference pressure profile $p_0(z)$, and $T_0(z)$ is the temperature of the reference profile. Using the ideal gas law to substitute for T_0 gives an expression for the reference density,

$$\rho_0 = \frac{p_0}{R\theta_0}\left(\frac{p_r}{p_0}\right)^\kappa,$$

where R is the gas constant of air. This allows us to evaluate the $\partial\rho_0/\partial z$ in equation (9), again using hydrostatic balance of the reference state, to give

$$\frac{Dw}{Dt} = -\frac{\partial \Phi}{\partial z} - \frac{(\kappa-1)pg}{p_0} - \frac{g\rho}{\rho_0}. \quad (10)$$

The (total) potential temperature can be written $\theta_{\text{tot}} = p_r^\kappa R^{-1}(p_{\text{tot}})^{1-\kappa}(\rho_{\text{tot}})^{-1}$ which we Taylor expand to first order assuming small perturbations θ, p and ρ to give

$$\theta = \theta_0\left[\frac{(1-\kappa)p}{p_0} - \frac{\rho}{\rho_0}\right]. \quad (11)$$

Substituting (11) into (10) gives

$$\frac{Dw}{Dt} = -\frac{\partial \Phi}{\partial z} + b,$$

where $b = g\theta/\theta_0$ is the buoyancy which according to the thermodynamic equation (6) is conserved following the flow:

$$\frac{Db}{Dt} = 0.$$

Lastly we substitute into the mass continuity equation (5) to give

$$\frac{\partial(\rho_0+\rho)}{\partial t} + \nabla\cdot[(\rho_0+\rho)\mathbf{u}] = 0.$$

Assuming that density perturbations can be neglected relative to ρ_0 (and disallowing sound waves) gives

$$\nabla\cdot(\rho_0\mathbf{u}) = 0.$$

We have arrived at the anelastic equations:

$$\frac{Du}{Dt} = -\frac{\partial\Phi}{\partial x}, \tag{12}$$

$$\frac{Dv}{Dt} = -\frac{\partial\Phi}{\partial y}, \tag{13}$$

$$\frac{Dw}{Dt} = -\frac{\partial\Phi}{\partial z} + b, \tag{14}$$

$$\frac{Db}{Dt} = 0, \tag{15}$$

$$0 = \nabla\cdot(\rho_0\mathbf{u}). \tag{16}$$

Boussinesq approximation

The anelastic version of the mass continuity equation (16) may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{w}{\rho_0}\frac{\partial\rho_0}{\partial z} = 0.$$

If we make the additional assumption that the vertical length scale of the circulation is much smaller than the vertical length scale over which $\rho_0(z)$ varies ($\sim$ 10km), we can replace the anelastic continuity equation with $\nabla\cdot\mathbf{u} = 0$ (i.e., incompressible flow). This gives the form of the Boussinesq equations that we use in class:

$$\frac{Du}{Dt} = -\frac{\partial\Phi}{\partial x}, \tag{17}$$

$$\frac{Dv}{Dt} = -\frac{\partial\Phi}{\partial y}, \tag{18}$$

$$\frac{Dw}{Dt} = -\frac{\partial\Phi}{\partial z} + b, \tag{19}$$

$$\frac{Db}{Dt} = 0, \tag{20}$$

$$0 = \nabla\cdot\mathbf{u}. \tag{21}$$