Internal gravity waves\textsuperscript{1}

In most places, and at most times, the atmosphere is stably stratified to unsaturated displacements. Here we consider what happens when a stably stratified fluid is perturbed. These introductory notes cover the simplest case of a Boussinesq fluid.

Boussinesq flow

We begin with the Boussinesq equations:

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{\partial \phi}{\partial x} \\
\frac{Dv}{Dt} &= -\frac{\partial \phi}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{\partial \phi}{\partial z} + b \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{Db}{Dt} &= 0
\end{align*}
\]

where \( u = (u, v, w) \) is the velocity, \( b \) is the buoyancy, \( \phi \) is the perturbation pressure divided by the reference density, and the Lagrangian derivative is \( D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla \). Note that the third of (1) becomes the equation of hydrostatic balance when \( dw/dt \) is negligible. We will replace this equation by

\[
\alpha \frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} + b .
\]

The constant \( \alpha \) is a trick: \( \alpha = 1 \), of course, but we shall carry it through the analysis so that we can, after the fact, look at the hydrostatic case by setting \( \alpha = 0 \).

Waves on a motionless basic state

Assume a motionless, stratified, basic state, with \( u_0 = v_0 = w_0 = 0, b_0 = N^2 z \) + constant, \( \phi_0 = \int b_0 \, dz \). \( N^2 > 0 \), so this state is stably stratified. Then we assume there are small-amplitude perturbations to the basic state denoted \( u', v', w', b' \) and \( \phi' \) such that \( b = b_0 + b' \) and similarly for the other variables. The perturbations

\textsuperscript{1}These notes are adapted from notes courtesy of Alan Plumb
approximately satisfy the linearized equations resulting from the neglect of nonlinear terms such as $u'^2 \frac{\partial u'}{\partial x}$ in (1):

\[
\begin{align*}
\frac{\partial u'}{\partial t} &= -\frac{\partial \phi'}{\partial x} \\
\frac{\partial v'}{\partial t} &= -\frac{\partial \phi'}{\partial y} \\
\alpha \frac{\partial w'}{\partial t} &= -\frac{\partial \phi'}{\partial z} + b' \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 \\
\frac{\partial b'}{\partial t} + N^2 w' &= 0
\end{align*}
\]

Denoting the real part by $\text{Re}$, look for wavelike solutions of the form

\[
\begin{pmatrix}
u' \\
v' \\
w' \\
\phi' \\
b'
\end{pmatrix} = \text{Re} \begin{pmatrix}
U \\
V \\
W \\
\Phi \\
B
\end{pmatrix} e^{i(kx+ly+mz-\omega t)}
\]

Then

\[
\begin{align*}
\omega U - k\Phi &= 0 \\
\omega V - l\Phi &= 0 \\
\omega \alpha W - m\Phi - iB &= 0 \\
kU + lV + mW &= 0 \\
-i\omega B + N^2 W &= 0
\end{align*}
\]

From the last of these, $iB = N^2 W/\omega$, so the third eq. gives $(\omega \alpha - N^2/\omega) W - m\Phi = 0$. Substitute for $U, V, W$ from the first three equations into the fourth equation to give

\[
\left[ \frac{k^2}{\omega} + \frac{l^2}{\omega} + \frac{m^2}{(\omega \alpha - N^2/\omega)} \right] \Phi = 0 ,
\]

and hence

\[
\omega^2 = \frac{N^2 (k^2 + l^2)}{\alpha (k^2 + l^2) + m^2}.
\]

**Nonhydrostatic case** ($\alpha = 1$) For the general case, $\alpha = 1$, and the dispersion relation is

\[
\omega = \pm N \sqrt{\frac{k^2 + l^2}{k^2 + l^2 + m^2}} \quad (2)
\]
Note that this can be written
\[ \omega = \pm N \sin \gamma, \]
where \( \gamma \) is the angle the wavenumber vector \( \mathbf{k} = (k, l, m) \) makes with the vertical. So \( |\omega| \leq N \).

The phase speed in the direction of \( \mathbf{k} \) is given by
\[ c = \frac{\omega}{|\mathbf{k}|} \]
and the group velocity is
\[ \mathbf{c}_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) = \frac{\omega m}{k^2 + l^2 + m^2} \left[ \frac{km}{k^2 + l^2 + m^2}, \frac{lm}{k^2 + l^2 + m^2}, -1 \right]. \]

Note:

1. \( \mathbf{c}_g \cdot \mathbf{k} = 0 \) : group propagation is along the phase lines

2. From the continuity eq., \( \mathbf{k} \cdot \mathbf{u}' = 0 \) — the fluid motions are along the phase lines. (Note that this implies no advection of wave properties; e.g., since \( b' \) does not vary along lines of constant phase, \( \mathbf{u}' \cdot \nabla b' = 0 \). Hence the nonlinear advection terms we neglected on the grounds of small amplitude are in fact zero — a monochromatic plane internal gravity wave in a uniform medium is in fact a nonlinear solution to the problem!)

3. Note that point (2) implies that fluid motions are normal to \( \mathbf{k} \). So as \( \gamma \to \pi/2 \), the motions are vertical and \( \omega \to N \), the buoyancy frequency; as \( \gamma \to 0 \), the motions are horizontal (against which the stratification offers no resistance) and \( \omega \to 0 \).

4. Note that if all components of \( \mathbf{k} \) are real, \( \omega \leq N \): disturbances with \( \omega > N \) cannot propagate.

5. \( (c_g)_x = m^2 k^2 c_x / [(k^2 + l^2 + m^2)(k^2 + l^2)] \), so the \( x \) components of phase and group velocities are in the same direction. Similarly, the \( y \) component. But \( (c_g)_z = -m^2 c_z / (k^2 + l^2 + m^2) \) — the vertical components of group and phase velocities have opposite signs.

So an upward (and rightward) propagating wave looks as shown in the following figure:
From a localized source oscillating with a single frequency $\omega$, the waves form rays (the “St Andrews’ cross”) at angles $\gamma = \sin^{-1}(\omega/N)$ to the horizontal, with the phase propagation across the rays:

**Hydrostatic case** ($\alpha = 0$) When $\alpha = 0$, the dispersion relation becomes

$$\omega = \pm \frac{N}{m} \sqrt{k^2 + l^2} = \pm N \tan \gamma$$

There is no longer any restriction $\omega \leq N$, so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts $\omega \gtrsim N$, but it should be good for $\gamma \ll 1$ ($\omega \ll N$). Equivalently, it requires $k^2 + l^2 \ll m^2$, i.e., vertical scales much less than horizontal scales.